A New Intersection Theorem in Topological Spaces with the Application in Game Theory

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Abstract- In this paper, a new intersection theorem for weakly transfer compactly closed valued mappings is established in topological spaces. As applications, new variational inequalities, a fixed point theorem and a maximal element theorem are obtained in noncompact GFC-spaces. Our results unify, improve and generalize some known results in recent reference. Finally, equilibrium existence theorems for abstract economies and qualitative games in noncompact GFC-spaces are yielded.

Keywords- GFC-space; T-KKM Mapping; Variational Inequality; Fixed Point; Maximal Element; Abstract Economy; Equilibrium

I. INTRODUCTION


The aim of this paper is to establish a new intersection theorem for weakly transfer compactly closed valued mappings in topological spaces. As applications, new variational inequalities, a fixed point theorem and a maximal element theorem are obtained in noncompact GFC-spaces. Our results unify, improve and generalize some recent results in the reference therein. Finally, equilibrium existence theorems for abstract economies and qualitative games in non-compact GFC-spaces are yielded.

II. PRELIMINARIES

Let X be a nonempty set. We denote by \( X \times 2^X \) the family of all nonempty finite subsets of X and the family of all subsets of X, respectively, by \( \triangle_n \) the standard n-dimensional simplex with vertices \( e_0, \ldots, e_n \). Let X and Y be two topological spaces. We denote by \( C(X, Y) \) the class of single-valued continuous maps of X into Y. Following Khanh et al. [5,6], let X be a topological space, Y be a nonempty set and F a family of continuous mappings \( f: X \rightarrow Y \). Then a triple \( (X, Y, F) \) is said to be a GFC-space if for each \( N = \{y_0, \ldots, y_n\} \subset Y \), there is \( j_N: X \to D_n \) of the family F. Let \( D \subset Y \) and \( S: Y \to 2^Y \) be given. D is called an S-subset of Y if for each \( N = \{y_0, \ldots, y_n\} \subset Y \) and each nonempty subset \( \{y_{i_1}, \ldots, y_{i_k}\} \subset N \), we have

\[ j_N(D) \cap S(D) \]

where \( D_k \) is the face of \( D_n \) corresponding to \( \{y_{i_1}, \ldots, y_{i_k}\} \).

(1)

Let \( (X, Y, F) \) be a GFC-space, \( Z \) a topological space, \( F: Y \to 2^Y \) and \( T: X \to 2^Z \) two mappings. F is said to be a T-KKM mapping if for each \( N = \{y_0, \ldots, y_n\} \subset Y \) and each nonempty subset \( \{y_{i_1}, \ldots, y_{i_k}\} \subset N \), we have

\[ T(j_N(D)) \cap \bigcup_{i \in Y} F(y_i) \]

A mapping \( T: X \to 2^Z \) is said to have the generalized KKM property if for each T-KKM mapping \( F: Y \to 2^Z \), the family \( \{cl(F(y))\}_{y \in Y} \) has the finite intersection property. By KKM(\( X, Y, Z \)) we denote the class of all mappings \( T: X \to 2^Z \) which enjoy the generalized KKM property.

Now, we introduced the following definitions and lemmas.

Definition 2.1 Let \( (X, Y, F) \) be a GFC-space, \( Z \) a topological space, \( T: X \to 2^Z \) a mapping and \( g: 2^Z \to R \) a real number. A function \( f: X \times Y \to R \) \( f \cup \{1\} \) is said to be generalized \( g \)-T-GFC-diagonally quasiconcave (resp., quasiconvex) in \( y \) if for each \( N = \{y_0, \ldots, y_n\} \subset Y \), each nonempty subset \( \{y_{i_1}, \ldots, y_{i_k}\} \subset N \) and each \( z \in \bigcup_{i \in Y} f_j(x_i(z)) \), we have

\[ \min_{y_{i_1} \ldots y_{i_k}} f(y_i, z) \leq g(\text{resp.,} \max_{y_{i_1} \ldots y_{i_k}} f(y_i, z)) \geq g. \]

Remark 2.1 Definition 2.1 unifies and generalizes Definition 2.6 of Tang et al. [3], Definition 4.1 of Ding and Wang [4], Definition 1.2 of Wen [7], Definition 3.1 of Ding [8], Definition 2.5(3) and 2.6 of Kirk et al. [9], Definition 2.4 of Ding [10], Definition 2.1 of Wen [11], and the other corresponding definitions.

The following Lemma is obvious.

Lemma 2.1 Let \( (X, Y, F) \) be a GFC-space, \( Z \) a topological space, \( T: X \to 2^Z \) a mapping and \( g: 2^Z \to R \) a real number. Then a function \( f: X \times Y \to R \) is generalized \( g \)-T-GFC-diagonally quasiconcave (resp., quasiconvex) in \( y \) if and only if the mapping \( F: Y \to 2^Z \) defined by
\[ F(y) := \{ z \mid Z, f(y, z) \in K \} \]
\[ (\text{resp., } F(y)) := \{ z \mid Z, f(y, z) \in g \} \]
for each \( y \in Y \) is a \( T \)-KKM mapping.

**Remark 2.2** Lemma 2.1 unifies and generalizes Proposition 3.1 of Tang et al.[3], Lemma 4.1 of Ding and Wang[6], Lemma 1.1 of Wen[9], Lemma 3.1 of Ding[8], Lemma 2.7 of Kirk et al.[10] and Lemma 2.1 of Wen[11].

**Definition 2.2**[11] Let \( X \) be a nonempty set, \( Y \) a topological space and \( K \) a nonempty compact subset of \( Y \). A mapping \( G : X \rightarrow 2^X \) is said to be weakly transfer compactly open (resp., closed) valued relative to \( K \) if for each \( x \in X \), \( \{ y \in Y \mid f(x, y) \in \text{int}(G(y) \cap K)(\text{resp., } y \in Y \mid f(x, y) \in \text{cl}(G(y) \cap K)) \) implies that there exists \( x' \in X \) such that \( y \in \text{int}_Y(G(x') \cap K)(\text{resp., } y \in Y \mid \text{cl}_Y(G(x') \cap K)) \).

**Definition 2.3**[11] Let \( X \) be a nonempty set, \( Y \) a topological space, \( K \) a nonempty compact subset of \( Y \) and \( g \in R \) a real number. A function \( f : X \times Y \times R \rightarrow R \) is said to be weakly \( g \)-transfer compactly upper (resp., lower) semicontinuous (in short, \( g \)-t.c.u.s.c) relative to \( K \) in \( Y \) if for all \( x \in X \) and \( y \in Y \), \( f(x, y) \geq g \) (resp., \( f(x, y) \leq g \) ) implies that there exist a relatively open neighborhood \( N_Y \) of \( y \) in \( K \) and \( x' \in X \) such that \( f(x', y) \geq g \) (resp., \( f(x', y) \leq g \) ) for all \( z \in N_Y \).

**Lemma 2.2**[11] Let \( X \) be a nonempty set, \( Y \) a topological space, \( K \) a nonempty compact subset of \( Y \) and \( g \in R \) a real number. A function \( f : X \times Y \times R \rightarrow R \) is \( g \)-t.c.u.s.c relative to \( K \) if and only if the mapping \( F : X \rightarrow 2^Y \) defined by \( F(x) := \{ y \in Y \mid f(x, y) \in \text{int} G(y) \cap K \} \) (resp., \( F(x) := \{ y \in Y \mid f(x, y) \in \text{cl} G(y) \cap K \}) \) for each \( x \in X \) is weakly transfer compactly closed valued relative to \( K \).

The following lemma is the improving version of Lemma 2.1 of Ding et al.[2], Lemma 2.1 of Khanh et al.[5], Lemma 2.1 of Hai et al.[6] and Lemma 2.1 of Lin et al.[12].

**Lemma 2.3** Let \( X \) be a nonempty set, \( Y \) a topological space, \( K \) a nonempty compact subset of \( Y \) and \( F : X \rightarrow 2^Y \) a mapping. Then the following conditions are equivalent:

1. \( F \) is weakly transfer compactly closed (resp., open) valued relative to \( K \);
2. \( \bigcap_{i \in I} (K \cap F(x)) = \bigcap_{i \in I} (K \cap cl F(x)) \) and \( \bigcup_{i \in I} (K \cap int F(x)) \).

**Remark 3.1** If \( F \) is transfer compactly closed valued or transfer compactly open valued, then \( F \) is weakly transfer compactly closed values, of course. If \( Y = Z = K \) is one of a compact \( L \)-convex space, compact hyperconvex space or a compact \( G \)-convex space, and \( F \) is KKM mapping, then the conditions that \( F \) is a nonempty compact subset of \( Z \) has the finite intersection property and there exists \( N \in I \) such that \( \bigcap_{i \in I} cl F(z) \) is held trivially. Therefore, Theorem 3.1 improves and generalizes Theorem 2.2(1) of Ding[8], Corollary 2.6 of Kirk et al.[9], Theorem 3.2(i) of Ding[10], Theorem 3.1 of Wen[11], Theorem 2.2 of Wen[13], Theorem 1 of Park[14], and Theorem 1.1 of Chowdhury et al.[15].

**Theorem 3.1** Let \( Y \) be a nonempty set, \( Z \) a topological space, \( K \) a nonempty compact subset of \( Z \) and \( F : Y \rightarrow 2^Z \) a mapping with weakly transfer compactly closed values relative to \( K \). Suppose that the family \( \{ cl F(y) \}_{i \in I} \) has the finite intersection property and there exists \( N \in I \) such that \( \bigcap_{i \in I} cl F(z) \) is held trivially. Then \( \bigcap_{i \in I} cl F(z) \) is nonempty compact subset of \( Z \) and \( F \) is KKM mapping, and \( \bigcap_{i \in I} cl F(z) \) is compact. In virtue of Lemma 2.3, we have

\[ \bigcap_{i \in I} F(y) = \bigcap_{i \in I} (K \cap \bigcap_{i \in I} F(y)) \]
\[ = \bigcap_{i \in I} (K \cap cl F(y)) \]
\[ = \bigcap_{i \in I} cl F(y) \]

Therefore,

\[ K \cap \bigcap_{i \in I} F(y) = \bigcap_{i \in I} (K \cap cl F(y)) \]
\[ = \bigcap_{i \in I} cl F(y) \]

**Theorem 3.2** Let \( Y \) be a nonempty set, \( Z \) a topological space, \( K \) a nonempty compact subset of \( Z \) and \( f : Y \times Z \rightarrow R \) a real number, and \( f : Y \times Z \rightarrow R \) a function such that
(1) \( \{c|_{\mathcal{Z}} \mathcal{Z} : f(y,z) \in \mathcal{G}\}_{\pi} \) has the finite intersection property;

(2) \( f(y,z) \) is w.g -t.c.l.s.c. relative to \( K \) in \( z \);

(3) there exists \( N \tilde{\mathcal{I}} \prec_\prec \wedge \) such that

\[
\bigcap_{\mathcal{I} \wedge} \{c|_{\mathcal{Z}} \mathcal{Z} : f(y,z) \in \mathcal{G} \} \subset K .
\]

Then there exists \( z^* \tilde{\mathcal{I}} \) \( K \) such that \( f(y,z^*) \notin \mathcal{G} \) for all \( y \in Y \).

**Proof** Define a mapping \( F : Y \to 2^Y \) by

\[
F(y) = \{ z \in \mathcal{Z} : f(y,z) \in \mathcal{G} \}, y \in Y .
\]

Then by (1), \( \{c|_{\mathcal{Z}} F(y)\}_{\pi} \) has the finite intersection property. By (2) and Lemma 2.2, \( F \) is weakly transfer compactly closed valued relative to \( K \). By (3), there exists \( N \tilde{\mathcal{I}} \prec_\prec \wedge \) such that \( \bigcap_{\mathcal{I} \wedge} c|_{\mathcal{Z}} F(y) \subset K \). In virtue of Theorem 3.1, we have

\[
K \supseteq \bigcap_{\mathcal{I} \wedge} F(y) .
\]

Take \( z^* \tilde{\mathcal{I}} \bigcap_{\mathcal{I} \wedge} F(y) \). Then \( z^* \tilde{\mathcal{I}} K \) and \( f(y,z^*) \notin \mathcal{G} \) for all \( y \in Y \).

**Remark 3.2** As shown in Remark 3.1, Theorem 3.2 unifies, improves and generalizes Theorem 3.2 of Wen[7], Theorem 3.5 of Ding[8], Theorem 2.8 of Kirk et al.[9], Theorem 4.2 of Ding[10], Theorem 2.1 and Theorem 2.2 of Chowdhury et al.[15].

**Theorem 3.3** Let \( (X,Y,F) \) be a GFC-space, \( Z \) a topological space, \( Y \) a nonempty subset of \( Z \), \( K \) a nonempty compact subset of \( Z \), \( T \tilde{\mathcal{I}} \) \( \mathcal{K} \mathcal{M}(X,Y,Z) \), \( g \tilde{\mathcal{I}} \) \( \mathcal{R} \) a real number and \( f : Y \times \mathcal{Z} \to \mathcal{R} \) a function such that

(1) for each \( z \in \mathcal{Z} \), \( \{ y \in Y : f(y,z) \in \mathcal{G} \} \) is empty or a \( r^{-1} \)-subset of \( \mathcal{Z} \);

(2) for each \( y \in Y \), \( f(y,z) \notin \mathcal{G} \);

(3) \( f(y,z) \) is w.g -t.c.l.s.c. relative to \( K \) in \( z \);

(4) there exists \( N \tilde{\mathcal{I}} \prec_\prec \wedge \) such that

\[
\bigcap_{\mathcal{I} \wedge} c|_{\mathcal{Z}} \mathcal{Z}: f(y,z) \in \mathcal{G} \} \subset K .
\]

Then there exists \( z^* \tilde{\mathcal{I}} K \) such that \( f(y,z^*) \notin \mathcal{G} \) for all \( y \in Y \).

**Proof** We claim that \( f(y,z) \) is generalized \( g \)-T-GFC-diagonally quasiconcave in \( y \). Otherwise, there exist \( N \tilde{\mathcal{I}} \prec_\prec \wedge \), \( y \in Y \), nonempty subset \( \{ y_0, \ldots, y_n \} \) \( \subset N \) and \( \tilde{\mathcal{I}} \) \( T(y \tilde{\mathcal{I}} N) \) such that \( \min_{y \in Y} f(y,z) \notin \mathcal{G} \) and then,

\[
\{ y_0, \ldots, y_n \} \mathcal{I} \mathcal{Y} \{ y \in Y : f(y,z) \in \mathcal{G} \} .
\]

By (1), we have,

\[
j \tilde{\mathcal{I}} (D) \cap (\{ y \in Y : f(y,z) \in \mathcal{G} \} ,
\]

so that,

\[
\{ y \in Y : f(y,z) \in \mathcal{G} \} .
\]

Hence, \( f(y,z) \notin \mathcal{G} \), which contradicts (2).

Now, define a mapping \( F : Y \to 2^Y \) by

\[
F(y) = \{ z \in \mathcal{Z} : f(y,z) \in \mathcal{G} \}, y \in Y .
\]

Note that \( f(y,z) \) is generalized \( g \)-T-GFC-diagonally quasi-concave in \( y \). Hence, \( F \) is a \( T \)-KMM mapping by Lemma 2.1. Since \( T \tilde{\mathcal{I}} \mathcal{K} \mathcal{M}(X,Y,Z) \), then, \( \{c|_{\mathcal{Z}} F(y)\}_{\pi} \) \( \{c|_{\mathcal{Z}} Z : f(y,z) \in \mathcal{G} \} \) has the finite intersection property. Therefore, by (3), (4), in virtue of Theorem 3.2, there exists \( z^* \tilde{\mathcal{I}} K \) such that \( f(y,z^*) \notin \mathcal{G} \) for all \( y \in Y \).

**Theorem 3.4** Let \( (X,Y,F) \) be a GFC-space, \( Z \) a topological space, \( Y \) a nonempty subset of \( Z \), \( K \) a nonempty compact subset of \( Z \), \( T \tilde{\mathcal{I}} \) \( \mathcal{K} \mathcal{M}(X,Y,Z) \), \( s \tilde{\mathcal{I}} C(ZZ) \) a continuous map and \( p : Z \to \{ 1 \} \) \( \{ a \} \) a nonempty valued mapping such that

(1) \( P^i \) is weakly transfer compactly open valued relative to \( K \);

(2) there exists \( N \tilde{\mathcal{I}} \prec_\prec \wedge \) such that

\[
\bigcap_{\mathcal{I} \wedge} c|_{\mathcal{Z}} (s^{-1}(Y \setminus P^{-1}(y))) \subset K .
\]

(3) for each \( z \in \mathcal{Z} \), \( P(s(z)) \) is a \( T^i \)-subset of \( Y \).

Then there exists \( z^* \tilde{\mathcal{I}} Z \) such that \( z^* \tilde{\mathcal{I}} P(s(z^*)) \).

**Proof** Define a function \( f : Y \times \mathcal{Z} \to \mathcal{R} \) by

\[
f(y,z) = \begin{cases} 1, & \text{if } y \tilde{\mathcal{I}} P(s(z)), \\ 0, & \text{if } y \tilde{\mathcal{I}} P(s(z)).
\end{cases}
\]

for each \( (y,z) \in Y \times \mathcal{Z} \).

Suppose the conclusion is false, which implies that for each \( z \in \mathcal{Z} \), \( f(z,z) \notin \mathcal{G} \). By (1) and the continuity of \( s \), \( s^{-1} P^{-1} \) is weakly transfer compactly open valued relative to \( K \). Note that

\[
s^{-1} P^{-1}(y) = \{ z \in \mathcal{Z} : f(y,z) = 0 \} .
\]

Then \( f(y,z) \) is w.0-t.c.l.s.c relative to \( K \) in \( z \) by Lemma 2.2. By (2), there exists \( N \tilde{\mathcal{I}} \prec_\prec \wedge \) such that

\[
\bigcap_{\mathcal{I} \wedge} c|_{\mathcal{Z}} Z : f(y,z) = 0 \} \subset K .
\]

By (3), for each \( z \in \mathcal{Z} \), \( \{ y \in Y : f(y,z) = 0 \} \) is a \( T^i \)-subset of \( Z \). In virtue of Theorem 3.3, there exists \( z^* \tilde{\mathcal{I}} K \) such that \( f(y,z^*) \notin \mathcal{G} \) for all \( y \in Y \), which implies that \( P(s(z^*)) = \), which contradicts that \( P \) is nonvalued.

**Remark 3.3** Theorem 3.4 unifies, improves and generalizes Theorem 3.1 of Kirk et al.[9], Theorem 3.5 of Wen[10], Theorem 2.4 of Wen[11], Theorem 2, 3, 4, 8 of Park[14], and Theorem 2.3-A of Chowdhury et al.[15]. Lemma 2.2.
of Zhang\cite{16}, Lemma 1 of Wu\cite{17}, Corollary 2 and Corollary 3 of Chen and Shen\cite{18}.

As an immediate consequence of Theorem 3.4, we have the following existence theorem for maximal elements.

**Theorem 3.5** Let \((X,Y,F)\) be a GFC-space, \(Z\) a topological space, \(Y\) a nonempty subset of \(Z\), \(K\) a nonempty compact subset of \(Z\), \(T \subseteq \KKM(X,Y,Z)\), \(s \subseteq C(Z, Z)\) a continuous map and \(P : Z \otimes 2^X\) a nonempty valued mapping such that

1. \(P^1\) is weakly transfer compactly open valued relative to \(K\);
2. there exists \(N \subseteq <Y>\) such that
   \[
   \bigcap_{\delta Y} \text{cl}_\delta(\varepsilon^1(Z \cap P^1(\delta)(Y)) \cap K;
   
3. for each \(z \in I\), \(P(s(z))\) is a \(T^1\)-subset of \(Y\);
4. for each \(z \in I\), \(z \in P(s(z))\).

Then, there exists \(z \in I\) \(Z\) such that \(P(z) = \).

**IV. APPLICATION IN GAME THEORY**

In this section, we shall establish the equilibrium existence theorems for abstract economies and qualitative games in GFC-spaces. Following Wen\cite{19}, let \(I\) be a set of agents (players). An abstract economy (generalized game) \(E := (X_i; A_i, B_i, P_i)_{i \in I}\), is defined as a family of ordered quadruples \((X_i; A_i, B_i, P_i)\), where for each \(i \in I\), \(X_i\) is a nonempty set (choice set or strategy set), \(A_i, B_i; X := P_{\beta_i}X_i 2^X\) are constraint mappings and \(P_i; X \otimes 2^X\) is a preference mapping. An equilibrium for \(E\) is a point \(z \in X\) such that for each \(i \in I, \tilde{x}_i = p_{\tilde{y}}(\tilde{x}_i) = \text{cl}_\delta B_{\tilde{y}}(\tilde{x}_i)\) and \((A \cap P)(\tilde{x}_i)\) is a finite index set, for each \(i \in I, X_i\) is a nonempty set (strategy set) and \(P_i; X := P_{\beta_i}X_i 2^X\) is a preference mapping. A point \(z \in X\) is said to be an equilibrium for the qualitative game \(G\) if for each \(i \in I, P(\tilde{x}_i) = \).

**Theorem 4.1** Let \(X, Y\) be topological spaces, \((X,Y,F)\) be a GFC-space, \(K\) a nonempty compact subset of \(Y\), \(T \subseteq \KKM(X,Y,Z)\), \(A, B; Y \otimes 2^Y\) \(\subseteq \) two constraint mappings, \(P; Y \otimes 2^Y\) a preference mapping, \(E := (X; A, B, P)\) an abstract economy and \(F := (Y; A, B, P)\) an abstract economy such that

1. \(A^1, B^1\) and \(P^1\) are weakly transfer compactly open valued relative to \(K\);
2. \(F\) is compactly open;
3. there exists \(N \subseteq <Y>\) such that
   \[
   \bigcap_{\delta Y} \text{cl}_\delta(\varepsilon^1(Y \cap (A^1 \cap P^1(\delta)(Y)) \cap K;
   
4. for each \(y \in I\), \(A(y), B(y)\) and \(P(y)\) are \(T^1\)-subsets of \(Y\);
5. for each \(y \in I\), \(y (A \cap P)(y)\).

Then, \(E\) has an equilibrium in \(Y\).

**Proof** Define a mapping \(T : Y \otimes 2^Y\) by
\[
T(y) = \begin{cases} (\hat{A} \cap P)(y), & \text{if } y \in I Y \setminus F, \\ B(y), & \text{if } y \in I Y, \end{cases}
\]
for each \(y \in I Y\). Then \(T\) is nonempty valued and for each \(z \in I Y\),
\[
T^{-1}(z) = \{y \in I Y \mid (y T \cap (A \cap P)(z)) \cap (Y \cap (B \cap Y^{-1}(z))
\]
By (1) and (2), \(T^1\) is weakly transfer compactly open valued relative to \(K\). Note that for each \(z \in I Y\),
\[
Y \setminus T^{-1}(z) = (Y \setminus (A \cap P(z))) \cap (Y \setminus (B \cap Y^{-1}(z))
\]
By (3), there exists \(N \subseteq <Y>\) such that
\[
\bigcap_{\delta Y} \text{cl}_\delta(\varepsilon^1(Y \setminus T^{-1}(\delta)(Y)) \cap K
\]
By (4), for each \(y \in I Y\), \(T(y)\) is a \(T^1\)-subset of \(Y\). In virtue of Theorem 3.4, there exists \(y \in I Y\) such that \(y \in T(y)\). But by (5), \(y \in I (A \cap P)(y)\). Thus, \(y \in (A \cap P)(y)\) \(\subseteq \). Therefore, \(y \in (A \cap P)(y)\).

**Theorem 4.2** Let \(I\) be a finite index set, for each \(i \in I\), \(X_i, Y_i\) be topological spaces, \(P_i; Y_i \otimes 2^Y\) be a preference mapping, \(K\) a nonempty compact subset of \(Y\), \(X := P_{\beta_i}X_i 2^X\), \((X,Y,F)\) be a GFC-space, \(T \subseteq \KKM(X,Y,Z)\). Suppose the qualitative game \(G := (X, P, \beta_i)_{i \in I}\), such that

1. for each \(i \in I\), \(P_i^{-1}\) is weakly transfer compactly open valued relative to \(K\);
2. there exists \(y^* \in I Y\) such that for each \(i \in I\),
   \[
   \text{cl}_\delta(Y \cap (P_i^{-1}(y^*)(y)) \subseteq K
   
3. for each \(i \in I\) and each \(y \in I Y\), \(P_i^{-1}(y)\) is empty or \(T^1\)-subset of \(Y\);
4. for each \(y \in I Y\), there exists \(i \in I\) such that \(y \in P_i^{-1}(y)\).
Then, \( G \) has an equilibrium in \( Y \).

**Proof** Let
\[
I(y) = \{ \text{g} \mid J(y) \}	ext{, } y \in Y,
\]
\[
J(y) = \{ \text{g} \mid J(I_p(y)) \}	ext{, } y \in Y
\]
and define a mapping \( P: Y \to 2^Y \) by
\[
P(y) = \bigcap_{i \in I} P^{-1}_i P_i(y), \text{ if } I(y) = \emptyset,
\]
\[
\{ \} , \text{ if } I(y) \neq \emptyset.
\]
for each \( y \in Y \). Then, \( P(y) \) is and only if \( I(y) \neq \emptyset \). For each \( y \in Y \), if \( I(y) = \emptyset \), then \( y \) is an equilibrium of \( G \) in \( Y \). If \( I(y) \neq \emptyset \), then \( P(y) = \bigcap_{i \in I} P^{-1}_i P_i(y) \) and for each \( z \in Y \),
\[
P^{-1}(z) = \{ y \mid J(y) = z \}
\]
\[
= \{ y \mid J(y) = \bigcap_{i \in I} P^{-1}_i P_i(y) \}
\]
\[
= \{ y \mid J(y) = P^{-1}_i P_i(y), i \}
\]
\[
= \{ y \mid J(y) = P^{-1}_i P_i(y), i \}
\]
\[
= \bigcap_{i \in I} P^{-1}_i P_i(y)
\]
By (1), \( P^{-1} \) is weakly transfer compactly open valued relative to \( K \). By (2), there exists \( N = \{ * \} \) such that
\[
\bigcap_{i \in J(z)} c_{l_i}(Y \setminus P^{-1}(y)) = c_{l_i}(Y \setminus P^{-1}(y))
\]
\[
= c_{l_i}(Y \setminus P^{-1}(y))
\]
\[
= \bigcup_{i \in J(z)} c_{l_i}(Y \setminus P^{-1}(y))
\]
\[
= \bigcap_{i \in I} P^{-1}_i P_i(y)
\]
By (3), for each \( y \in Y \), \( P(y) \) is a \( T^1 \)-subset of \( Y \). By (4), for each \( y \in Y \), \( P(y) \). Hence, in virtue of Theorem 3.5, there exists \( \hat{y} \in Y \) such that \( P(\hat{y}) = \emptyset \), which implies that \( I(\hat{y}) = \emptyset \), which in turn implies that \( P(\hat{y}) = \emptyset \) for all \( i \in I \). Therefore, \( \hat{y} \) is an equilibrium of the qualitative game \( G \) in \( Y \).

**Theorem 4.3** Let \( I \) be a finite index set, for each \( i \in I \), \( X_i, Y_i \) be topological spaces, \( A_i, B_i : Y_i = P_{A_i} Y_i \to 2^{X_i} \) be constraint mappings, \( F_i : Y \to 2^{X_i} \) be a preference mapping, \( F = \{ \text{g} \mid J(p_i(y)) \} \subset X = P_{A_i} X_i, (X, Y, F) \) be a GFC-space, \( K \) a nonempty compact subset of \( Y \), \( T \) a KKM(\( X, Y, Z \)). Suppose the abstract economy \( E = (X; A_i, B_i, P_i)_{i \in I} \), such that

(1) for each \( i \in I \), \( A_i, P_i \) are weakly transfer compactly open valued relative to \( K \);

(2) for each \( i \in I \), \( F_i \) is compactly open;

(3) there exists \( y^* \in Y \) such that for each \( i \in I \), \( y \in A_i \setminus p_i(y^*) \), \( K \) and \( Y \setminus (F_i \setminus p_i(y^*)) \), \( K \);

(4) for each \( i \in I \) and each \( y \in Y \), \( p_i^{-1} p_i(y) \) and \( p_i^{-1} A_i(y) \) are empty or \( T^1 \)-subset of \( Y \);

(5) for each \( y \in Y \), there exists \( \hat{y} \in I \) such that \( y \in p_i^{-1} A_i(y) \);

(6) for each \( i \in I \) and each \( y \in Y \), \( A_i(y) \).

Then, \( E \) has an equilibrium in \( Y \).

**Proof** For each \( i \in I \), define a mapping \( Q_i : Y \to 2^Y \) by
\[
Q_i(y) = \bigcap_{i \in I} (A_i \setminus p_i(y)), \text{ if } y \in Y, \{ \}
\]
\[
\text{if } y \in \{ \}
\]
\[
\text{for each } y \in Y \). Then, for each \( y \in Y \),
\[
Q_i^{-1}(y) = \{ y \mid J(y) = Q_i(y) \}
\]
\[
= \{ y \mid J(y) = A_i \}
\]
\[
= \bigcup_{i \in I} (A_i \setminus p_i(y)) \}
\]
\[
= (F_i \setminus A_i(y)) \}
\]
\[
= A_i(y) \}
\]
By (1) and (2), for each \( i \in I \), \( Q_i^{-1} \) is weakly transfer compactly open valued relative to \( K \). By (3), there exists \( y^* \in Y \) such that for each \( i \in I \),
\[
Y \setminus Q_i^{-1}(y^*) \}
\]
\[
= (Y \setminus A_i(y^*)) \}
\]
\[
= \bigcap_{i \in I} (A_i \setminus p_i(y^*)) \}
\]
By (4), for each \( i \in I \) and each \( y \in Y \), \( p_i^{-1} Q_i(y) \) is empty or \( T^1 \)-subset of \( Y \). By (5), for each \( i \in I \), \( y \in Y \), there exists \( \hat{y} \in I \) such that \( y \in p_i^{-1} Q_i(y) \). Hence, in virtue of Theorem 4.2, there exists \( \hat{y} \in Y \) such that \( Q_i(y) \) for all \( i \in I \). In the other hand, by (6), for each \( i \in I \), \( A_i(\hat{y}) \), and then \( A_i(\hat{y}) = \emptyset \), which implies that
\[
\hat{y} \in Y \setminus F_i \}
\]
\[
= (Y \setminus A_i(y^*)) \}
\]
\[
= \bigcap_{i \in I} (A_i \setminus p_i(y^*)) \}
\]
\[
\text{and hence } \hat{y} = p_i(\hat{y}) \}
\]
\[
= \bigcap_{i \in I} (A_i \setminus p_i(y^*)) \}
\]
\[
\text{Therefore, } \hat{y} \in Y \}
\]
\[
\text{is an equilibrium of the abstract economy } E \in Y\}
\]

**V. CONCLUSION**

In this paper, a new definition of generalized -T-GFC-diagonally quasiconcave (resp., quasiconvex) function is first introduced, and the property of generalized -T-GFC-dia-gonally quasiconcave (resp., quasiconvex) functions is studied. And then, a new intersection theorem for weakly generalized -transfer compactly closed values mappings is established.
in non-compact topological spaces. As applications, new variational inequalities for weakly $g$-transfer compactly lower semi-continuous functions are studied in noncompact topological spaces and noncompact GFC-spaces, and a fixed point theorem and a maximal element theorem for weakly $g$-transfer compactly open inverted valued mappings are obtained in noncompact GFC-spaces. These in turn are applied to yield equilibrium existence theorems for abstract economies and qualitative games with weakly $g$-transfer compactly closed valued constraint mappings and preference mappings in noncompact GFC-spaces. Our results unify, improve and generalize corresponding results in the reference cited above.

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