Abstract-In the information age, service products will be sold through online advance booking systems, and the reservations will be canceled by the end of pre-sale period. Therefore, in order to maximize the revenue, firms will adjust the selling price of service products dynamically. Taking into account both initial order quantity and dynamic pricing, this paper investigates three questions 1) \( n \), the number of price adjustments in the pre-sale period, 2) \( p_t \), the selling price which is set during time \([T_{1i}, T_i]\), 3) \( Q \), the initial ordering quantity of the service product. A mathematical model is developed to find the optimal order quantity, and a solution algorithm is proposed to determine the optimal dynamic pricing strategy. The numerical example shows the effectiveness of the model and the algorithm, and the superiority of dynamic pricing strategy compared with the static strategy. Additionally, the influence of other parameters on the optimal policies is analyzed when the times of price adjustment times are known. The innovation of this paper is to study the optimal number of dynamic pricing of service products under an online advance booking system.

Keywords-Service Products; Dynamic Pricing; Order Policies; Reservation

I. INTRODUCTION

Simultaneity of production and consumption is a main feature which distinguishes the service products from tangible products, that’s to say, production process and consumption process of service products are often carried out simultaneously. For example, for the passenger services of airlines, where transportation services (production process) and the services enjoyed by passengers (consumption process) occur and finish at the same time. This is the same to hotel accommodation: the check-in will announce the beginning of the consumption process as well as the production process, while check-out means that the consumption process and production process are all over. Demand of service products is susceptible to seasons, which will lead to large fluctuations in demand in the pre-sale period. Unlike the tangible products which can adjust the demand through inventory, service products cannot be stored. In order to maximize its revenue, sellers often adjust price dynamically according to the market demand in the pre-sale period. The emergence of online reservation systems by which customers can make reservation and also can cancel orders before the pre-sale period exerts negative impacts on service suppliers, therefore, dynamic pricing in the pre-sale period is the main way to stimulate demand, with the studies on revenue management and dynamic pricing gradually being enriched.

II. LITERATURE REVIEW

Kincaid W. M. and Darling D. A. [1] (1963) were the earliest researchers to study the continuous-time dynamic pricing of perishable goods. Gallego G and Van Ryzin G [2] (1994) connected dynamic pricing with revenue management firstly, which is the base of subsequent dynamic pricing articles. Chatwin R. E. [3] (1999) considered a continuous-time inventory problem that a retailer sets the price on a fixed number of a perishable asset that must be sold prior to its shelf life. The retailer can dynamically adjust the price within a finite number of allowable prices. Demand for the product is Poisson with an intensity that is inversely related to the price. The optimal policy is piecewise-constant. The maximum expected revenue is nondecreasing and concave in both the remaining inventory and the time.

Zhao W., Zheng Y. S. [4] (2000) supposed that customers’ arriving, according to a nonhomogeneous Poisson process, show that the optimal price decreases with the inventory at any given time and also identify a sufficient condition under which the optimal price decreases over time for a given inventory level. Guan Z Z, Huang S Z, Shi B S [5] (2005) proposed a continuous-time dynamic pricing model with customer reservation price, obtaining an optimal solution under the condition that demand is a Poisson process and reservation price is exponential distribution. Pan W., Wang S. Y., Hua G. W. et al. [6] (2010) thought that with the development of e-commerce, many “bricks-and-mortar” firms have transformed to “clicks-and-mortar” by incorporating online stores, in addition, they examine the optimal order policy and optimal price adjustment time for a “clicks-and-mortar” firm, and derive the optimal pricing strategy in every period. Qin J., Ni L. S., Miu L. X. [7] (2011) investigated marketing strategy of the seasonal products which consider the procurement quantity and the dynamic pricing simultaneously. Customers could purchase products on the spot or through internet, and the demands in both purchasing ways are assumed to be time and price dependent. A mathematical model is developed to find the optimal ordering quantity and dynamic pricing setting, whose objective is to
maximize the profit of retailers, and then an approach is proposed to determine the optimal decision. Shen C. L., Zhang X. X. [8] (2011) discussed the dynamic pricing problem of short-life-cycle products, a game theory process between a retailer and its consumers is analyzed and the optimal purchasing policies for consumers are determined, and the optimal dynamic pricing policy is introduced for the maximization of overall revenues finally. Li H., Xiong Z. K., Peng Z. Q. [9] (2011) introduced a dynamic pricing model for two retailers whose perishable products were sold out strategically to finite consumers. Li H., Xiong Z. K., Qu W. D. [10] (2011) considered that the airline divides the passengers into two categories by offering discount tickets, and controls the sales of discount tickets and implements dynamic pricing for ordinary tickets. Luo L., Peng J. H. [11] (2007) developed a continuous-time dynamic pricing model for two competitive flights with stochastic control theory and game theory. Two price levels are taken into account for each flight and have derived equilibrium solution. You P. S. [12] (1999) considered a multiple booking class airline-seat inventory control problem that relates to either a single flight leg or to multiple flight legs. You P. S. [13] (2004) considered that the firm sells products through an advance booking system, in order to maximize the total expected profit by optimizing product price. In addition, considering that demand is price-dependent and customers with reservations may cancel advance orders, it is necessary to develop a continuous time model to simultaneously determine the ordering quantity and selling prices. Li, H. and Y. Xiong [14] (2008) extended the GVR model to two airlines’ competition, and proved that the existence of Nash equilibrium. Currie, C. R., Cheng and H. Smith [15] (2008) studied the competition of two Airlines. Assuming that potential customers’ arrivals obey nonhomogeneous Poisson process, the arrival rate depends on the sales of the remaining time, the possibility of customers buying tickets between the two airlines depends on the remaining time and the two airlines’ selling prices. Zhang, Coope [16] (2009) considered two parallel flights dynamic pricing model based on the behaviour of customers. They used Markov decision process to establish a discrete-time stochastic dynamic model, and used the value-based heuristic approximation methods to solve the problems of high Markov, and finally the upper and lower bounds of the value function were obtained. Luo L., Xiao B. C. [17] (2012) studied the dynamic pricing strategy of two parallel airlines on the same route under a relative monopoly market for an airline corporation. With the stochastic control theory, dynamic pricing continuous time model was developed, the properties of the value function and the threshold point of the time for optimum control were discussed. Numerical examples showed that this control policy was not only implementation friendly, but also more profitable than the policy for independent demand.

However, the aforementioned literature reviews did not capture the situation that customers with reservations can make cancellations except for You P. S. [13] (2004) who, unfortunately, ignored the dynamic pricing. In reality, reservation systems are used by numerous service industries, customers may cancel their orders before availing of their services, and the sellers will adjust products’ prices dynamically. The literature review on dynamic pricing shows that most studies, either discrete time model or continuous time model, focused on the continuous price change as one factor, and no studies on price adjustment frequency in the pre-sale period were carried out. But in reality, the price will be adjusted several times according to the actual situation in the pre-sale period. For example, hotels will adjust selling prices according to the sales situation in the pre-sale period. The same situation also occurs in the airline corporation. Therefore, this paper tries to study the optimal number of price adjustment to maximize revenue. To simplify the problem, the pre-sale period is divided into n averaging sections, and the optimal ordering quantity in the pre-sale period and the optimal number n of price adjustment will be determined in this paper.

The rest of this paper is organized as follows. Section 3 specifies the model like the description and basic assumptions. In section 4, we construct a generic model from which the structural properties of the optimal policy are derived. The dynamic pricing model on demanding is a linear function as discussed in section 5. In section 6, numerical examples are used to study the impact of yield management on the expected revenue, and section 7 is the conclusions. Model description and basic assumption

III. MODEL DESCRIPTION AND BASIC ASSUMPTION

Considering a seller of service supplier sells products by online reservation systems in the pre-sale period, customers can pre-order their demands through this system, and can also cancel their order only by the end of pre-sale period at the sacrifice of a cancellation penalty fee. Meanwhile, in order to increase sales and maximize its revenue, the seller will adjust price dynamically. To simplify the problem, the pre-sale period is divided into n averaging sections; we establish a continuous time model by setting a set of discrete prices to study the optimal ordering quantity and dynamic pricing strategy. Several assumptions are given as follows.

a. The proportion of cancellations is assumed to be independent of the number of reservations and decreases with time. That is, the proportion of cancellations is

\[ \theta(t) = \frac{\eta}{t} \]

Where \( \eta \) is a positive constant with the value of \( (0, 1) \) and \( t \) indicates the time past since the start of the selling period. Although the assumption that cancellation rate decreases with time is restrictive, such a phenomenon is not uncommon because the number of customers who change their plans often decreases as the consumption day draws nearer.
b. Assuming unit penalty cost for cancellation \( r_i \) is a linear function of price \( p_i \).

\[
r_i = \lambda p_i, \; i = 1, 2, \ldots n
\]

Where \( \lambda \) is a positive constant with the value of \((0, 1)\), which is a proportion of price \( p_i \).

c. The pre-sale period \( L \) is divided into \( n \) equal stage time, and the price of product is different from each other. There exist a price collection \( p_i = \{ p_1, p_2, \ldots, p_n \} \), and the ordering quantity of the service product \( Q \) will be sold out at the end of time. The following notation is used throughout this study.

- \( T \) length of advance selling period;
- \( t \) time index, \( 0 \leq t \leq T \);
- \( n \) the number of price adjustments in the pre-sale period, a decision variable;
- \( p_i \) the selling price which is set during time \( [T_i, T_{i+1}] \), a decision variable;
- \( Q \) ordering quantity of the service product, a decision variable;
- \( c \) unit purchasing cost;
- \( r_i \) unit penalty cost for cancellation during time \([T_i, T_{i+1}]\);
- \( T_i \) setting price at the \((i+1)\) time, \( T_i = \frac{iL}{n} \);
- \( d(p_i) \) demand rate during [\( T_i, T_{i+1} \)];
- \( \theta(t) \) cancellation rate at time \( t \);
- \( I_i(t) \) reservation level during \([T_i, T_{i+1}]\).

IV. MODELING

According to the above, the reservation level of the system during time period \([T_i, T_{i+1}]\) can be given by the following differential equation:

\[
\frac{dI_i(t)}{dt} = d(p_i) - \theta(t) I_i(t), \quad T_i \leq t \leq T_{i+1}, \; i = 1, 2, 3, \ldots n
\]

As \( \theta(t) = \eta/t \), the reservation level is

\[
I_i(t) = t^\eta \frac{d(p_i)}{\eta+1}, \quad T_i \leq t \leq T_{i+1}, \; i = 1, 2, 3, \ldots n
\]

Where \( c_i \) is a constant. Clearly, the initial reservation level is 0, Thus, the boundary condition for the reservation level is \( I_1(0) = 0 \), so \( c_1 = 0 \).

\[
I_i(t) = \begin{cases} 
\frac{t}{\eta+1} d(p_i), & i = 1 \\
t^\eta \frac{d(p_i)}{\eta+1} + c_i, & T_i \leq t \leq T_{i+1}, \; i = 2, 3, \ldots n
\end{cases}
\]

A. Sales Revenue

Let \( n_i^i \) denote the expected sales volume within time \([T_i, T_{i+1}]\). Then, since the demand rate is \( d(p_i) \) and the cancellation rate is \( \theta(t) \), \( n_i^i \) is expressed as
\[ n_i^c = \int_0^{T_i} [d(p_i) - \theta(t) I_i(t)] dt = \frac{T_i}{\eta + 1} d(p_i) \] \tag{4} 

\[ n_i^c = \int_{T_i}^{T_{i+1}} (d(p_i) - \theta(t) I_i(t)) dt = \frac{T_{i+1} - T_i}{\eta + 1} (T_{i+1} - T_i) d(p_i) + c_i (T_{i+1} - T_i), \quad i = 2, 3, \ldots, n \] \tag{5} 

Consequently, the total sales revenues are given by

\[ R_s = n_i^c p_i + \sum_{i=2}^{n} n_i^c p_i = \frac{T_1}{\eta + 1} p_1 d(p_1) + \sum_{i=2}^{n} \left[ \frac{T_{i+1} - T_i}{\eta + 1} (T_{i+1} - T_i) p_i d(p_i) + (T_{i+1} - T_i) p_i c_i \right] \] \tag{6} 

**B. Cancellation Fee Revenues**

Let \( n_i^c \) denote the total expected number of cancellations during times \([T_i, T_{i+1}]\), then, we have

\[ n_i^c = \int_0^{T_i} \theta(t) I_i(t) dt = \frac{\eta}{\eta + 1} T_i d(p_i) \] \tag{7} 

\[ n_i^c = \int_{T_i}^{T_{i+1}} \theta(t) I_i(t) dt = \frac{\eta}{\eta + 1} (T_{i+1} - T_i) d(p_i) - c_i (T_{i+1} - T_i), \quad i = 2, 3, \ldots, n \] \tag{8} 

Cancellation fee revenues in the whole pre-sale period are presented by \( R_c = \sum_{i=1}^{n} n_i^c r_i \), because \( r_i = \lambda p_i, \lambda \in (0, 1) \), then

\[ R_c = n_1^c r_1 + \sum_{i=2}^{n} n_i^c r_i = \frac{\lambda \eta}{\eta + 1} T_1 p_1 d(p_1) + \sum_{i=2}^{n} \left[ \frac{\lambda \eta}{\eta + 1} (T_{i+1} - T_i) p_i d(p_i) - \lambda (T_{i+1} - T_i) p_i c_i \right] \] \tag{9} 

**C. Ordering Cost**

\( Q \) represents ordering quantity of the service product, then the total cost of ordering is

\[ C = Qc \] \tag{10} 

**D. Objective Function**

Let \( R(Q, p_i, n) \) denote the total profit that can be generated over period \([0, T]\), Eq. (11) then follows from Eqs. (6), (9) and (10).

\[ R(Q, p_i, n) = R_s + R_c - C 
= \frac{(1 + \lambda \eta) T_1}{\eta + 1} p_1 d(p_1) + \frac{1 + \lambda \eta}{\eta + 1} \sum_{i=2}^{n} (T_{i+1} - T_i) p_i d(p_i) + (1 - \lambda) \sum_{i=2}^{n} (T_{i+1} - T_i) p_i c_i - Qc \] \tag{11} 

As for the assumptions of \( c \), the ordering quantity of the service product \( Q \) will be sold out at the end of pre-sale period, that’s to say, the number of products ordered is consistent with the number of sales, \( Q = I_n(T_n) \), therefore, to maximize the objective function is transformed to find the best \( n \) and \( p_i \) of (11).

**E. Analysis**

The reservation level \( I_i(t) \) is a continuous function, then, \( I_{i+1}(T_i) = I_i(T_i) \), therefore,

\[ T_{i+1}^{\eta+1} \left[ \frac{T_{i+1}^{\eta+1}}{\eta + 1} d(p_i) + c_i \right] = T_i^{\eta+1} \left[ \frac{T_i^{\eta+1}}{\eta + 1} d(p_i) + c_i \right] \] \tag{12} 

From (12), we can obtain

\[ c_i = \frac{1}{\eta + 1} (d(p_{i-1}) - d(p_i)) T_{i+1}^{\eta+1} + c_{i-1}, \quad i = 2, 3, \ldots, n \] \tag{13} 

Knowing \( c_1 = 0 \), substituting (13), we can get,
\[ c_i = \frac{1}{\eta + 1} \sum_{j=2}^{j} (d(p_{j-1}) - d(p_j)) T_{ji}^{\eta+1}, \]
\[ = \frac{1}{\eta + 1} \left[ -T_{ji}^{\eta+1} d(p_i) + \sum_{j=1}^{j} \left( T_{ji}^{\eta+1} - T_{ji}^{\eta+1} \right) d(p_j) \right], \quad i = 2, 3, \ldots, n \]  

Knowing \( T_i = \frac{iL}{n} \), substituting (14), we can get,
\[ c_i = \frac{1}{\eta + 1} \left( \frac{L}{n} \right)^{\eta+1} \left[ -(i-1)^{\eta+1} d(p_i) + \sum_{j=1}^{j} \left( j^{\eta+1} - (j-1)^{\eta+1} \right) d(p_j) \right], \quad i = 2, 3, \ldots, n \]  

Meanwhile \( Q = I_n(T_n) \), substituting (3), we can get,
\[ Q = T_a^{-\eta} \left[ T_n^{\eta+1} d(p_n) + c_n \right], \quad n \geq 2 \]  

Formula (15) and \( T_i = \frac{iL}{n} \), substituting (16), we can get,
\[ Q = \frac{L n^{-\eta}}{(\eta + 1)n} \sum_{j=1}^{n} \left( (j)^{\eta+1} - (j-1)^{\eta+1} \right) d(p_j) \]  

Then,
\[ R(n, p_i) = \frac{L}{(\eta + 1)n} \left[ (1 + \lambda \eta) p_i d(p_i) + (1 + \lambda \eta) \sum_{j=2}^{n} p_j d(p_j) + (1 - \lambda) \sum_{j=2}^{n} u_j v_p d(p_j) \right] \]
\[ - \frac{L}{(\eta + 1)n} \left[ (1 - \lambda) \sum_{j=2}^{n} u_j p_i \sum_{j=2}^{n} w_j d(p_j) + an^{-\eta} \sum_{j=2}^{n} w_j d(p_j) \right] \]  

Where \( w_j = j^{\eta+1} - (j-1)^{\eta+1}, u_i = (i-1)^{-\eta} - i^{-\eta}, v_i = (i-1)^{\eta+1} \). Therefore, the mathematical optimization model of the dynamic pricing service product is described as follows,
\[ \max \quad R(n, p_i) \]
\[ \text{s.t.} \quad d(p_i) > 0 \]  

Theorem 1. For any given \( n \), if \( \frac{\partial^2 R}{\partial p_{k}^2} < - \sum_{j=1}^{n} \frac{\partial^2 R}{\partial p_{k} \partial p_{j}} \), the profits function \( R(n, p_i) \) is a concave function of \( p_i (i = 1, 2, \ldots, n) \).

Prove.
\[ \frac{\partial R}{\partial p_{k}} = \begin{cases} \frac{L}{(\eta + 1)n} \left[ (1 + \lambda \eta) d(p_1) + p_1 d'(p_1) \right] - \left[ (1 - \lambda) \sum_{i=2}^{n} u_i p_i + an^{-\eta} \right] d'(p_i), & k = 1 \\ \frac{L}{(\eta + 1)n} \left[ ((1 + \lambda \eta) + (1 - \lambda) u_k v_k) d(p_k) + p_k d'(p_k) \right] \\ - \frac{L}{(\eta + 1)n} \left[ an^{-\eta} w_k d(p_k) + (1 - \lambda) u_k \sum_{j=1}^{k-1} w_j d(p_j) \right], & k = 2, 3, \ldots, n \end{cases} \]
\begin{equation}
\frac{\partial^2 R}{\partial p_k \partial p_j} = \begin{cases}
\frac{L}{(\eta+1)n} \left[ (1+\lambda \eta) (2d(p_i) + p_i d'(p_i)) - \left( (1-\lambda) \sum_{i=2}^{n} u_i p_i + an' \right) d'(p_i) \right], & k=1 \\
\frac{L}{(\eta+1)n} \left[ (1+\lambda \eta) + (1-\lambda) u_i v_i \right] \left( 2d(p_i) + p_i d'(p_i) \right) - an' w_i d'(p_i), & k=2,3,\ldots n
\end{cases}
\end{equation}

(21)

Given \( f(n, p_i) = -R(n, p_i) \), then,

\[ a_{ik} = \frac{\partial^2 f}{\partial p_i^2} = -\frac{\partial^2 R}{\partial p_i \partial p_j}, \quad a_{ij} = \frac{\partial^2 f}{\partial p_i \partial p_j} = -\frac{\partial^2 R}{\partial p_i \partial p_j}, \quad k = 1, 2, \ldots n; k \neq j \]

Because \( d'(p_i) < 0, 0 < \lambda < 1 \), from (22), we can get \( \frac{\partial^2 R}{\partial p_i \partial p_j} > 0 \). \( a_{ij} = -\frac{\partial^2 R}{\partial p_i \partial p_j} < 0 \) then,

\[ |a_{ij}| = -a_{ij} = \frac{\partial^2 R}{\partial p_i \partial p_j}. \quad \text{As} \quad \frac{\partial^2 R}{\partial p_i \partial p_j} > 0, \quad \sum_{j=1, k \neq j}^{n} \frac{\partial^2 R}{\partial p_i \partial p_j} > 0. \quad \text{If} \quad \frac{\partial^2 R}{\partial p_k^2} < -\sum_{j=1, k \neq j}^{n} \frac{\partial^2 R}{\partial p_i \partial p_j}, \quad \text{then} \quad \frac{\partial^2 R}{\partial p_k^2} < 0. \quad a_{ik} > 0 \]

That’s to say, the diagonal elements of hessian matrix of \( f(n, p_i) \) are strictly greater than zero. Then \( |a_{ik}| = a_{ik} = -\frac{\partial^2 R}{\partial p_k^2} \).

When \( \frac{\partial^2 R}{\partial p_k^2} < -\sum_{j=1, k \neq j}^{n} \frac{\partial^2 R}{\partial p_i \partial p_j} \), then \( -|a_{ik}| = -\sum_{j=1, k \neq j}^{n} |a_{ij}| \Rightarrow |a_{ik}| > \sum_{j=1, k \neq j}^{n} |a_{ij}| \).

Therefore, the hessian matrix of function \( f(n, p_i) \) is strictly diagonally dominant matrix by rows.

To sum up, as \( \frac{\partial^2 R}{\partial p_k^2} < -\sum_{j=1, k \neq j}^{n} \frac{\partial^2 R}{\partial p_i \partial p_j} \), the hessian matrix of function \( f(n, p_i) \) is strictly diagonally dominant matrix and the diagonal elements are strictly greater than zero, according to disc theorem, the Eigen values of the hessian matrix are all positive which is a positive definite matrix. As \( f(n, p_i) = -R(n, p_i) \), the hessian matrix of the objective function \( R(n, p_i) \) is a negative definite matrix, that’s to say, For any given \( n \), the profits function \( R(n, p_i) \) is a concave function of \( p_i (i = 1, 2, \ldots, n) \).

\( R(n, p_i) \) is a concave function, so K-T condition is not only a necessary condition but also sufficient condition. Therefore, the points meeting K-T condition are the collection of the best prices which can maximize the objective function.

V. DEMAND AS A LINEAR FUNCTION OF THE DYNAMIC PRICING MODEL

Assume that the relationship between demand and price is linear and given \( d(p_i) = \alpha - \beta p_i \), where \( \alpha \) and \( \beta \) are positive constants.

a. When \( n=1 \), the optimal pricing is obtained:

\[ p = \frac{\alpha (1+\lambda \eta) + \beta a}{2\beta(1+\lambda \eta)} \]

(23)
And the optimal ordering quantity is

$$Q = \frac{L}{\eta+1}(\alpha - \beta p)$$  \hspace{1cm} (24)$$

b. When $$n \geq 2$$, by calculation, the optimal ordering quantity is

$$Q = \frac{Ln^{-\eta}}{(\eta + 1)n} \sum_{j=1}^{n} w_j (\alpha - \beta p_j)$$  \hspace{1cm} (25)$$

And the profit function can be

$$R(n, p_i) = \frac{L}{(\eta + 1)n} \left[(1 + \lambda \eta) p_i (\alpha - \beta p_i) + (1 + \lambda \eta) \sum_{i=2}^{n} p_i (\alpha - \beta p_i) + (1 - \lambda) \sum_{i=2}^{n} u_i v_p (\alpha - \beta p_i) \right]$$

$$- \frac{L}{(\eta + 1)n} \left[(1 - \lambda) \sum_{i=2}^{n} u_i p_i \sum_{j=1}^{i-1} w_j (\alpha - \beta p_j) + an^{-\eta} \sum_{j=1}^{n} w_j (\alpha - \beta p_j) \right], \hspace{0.5cm} i = 2,3,...,n$$  \hspace{1cm} (26)$$

Therefore, the optimal dynamic pricing and ordering policy model is

$$\max R(n, p_i)$$

$$s.t. \hspace{0.5cm} p_i < \frac{\alpha}{\beta}$$  \hspace{1cm} (27)$$

Considering the lagrangian function $$\Phi(n, p_i)$$ of $$R(n, p_i)$$ is as follows,

$$\Phi(n, p_i) = R(n, p_i) - \sum_{k=1}^{n} \mu_k \left(p_k - \frac{\alpha}{\beta} + Z_k^2\right), \hspace{0.5cm} \mu_k \geq 0$$  \hspace{1cm} (28)$$

Partial derivative of $$p_k, \mu_k, Z_k$$, we can get K-T necessary condition of $$R(n, p_i)$$.

$$\frac{\partial \Phi}{\partial p_k} = \frac{\partial R}{\partial p_k} - \mu_k = 0$$  \hspace{1cm} (29)$$

$$\frac{\partial \Phi}{\partial \mu_k} = -\left(p_k - \frac{\alpha}{\beta} + Z_k^2\right) = 0$$  \hspace{1cm} (30)$$

$$\frac{\partial \Phi}{\partial Z_k} = -2\mu_k Z_k = 0$$  \hspace{1cm} (31)$$

Because $$p_k < \frac{\alpha}{\beta}$$, from (30), we know that $$Z_k \neq 0$$, and then from (31), we can get $$\mu_k=0$$. According to (29), the necessary and sufficient conditions of $$R(n, p_i)$$ are $$\frac{\partial \Phi}{\partial p_k} = \frac{\partial R}{\partial p_k} = 0$$. Actually, for any given $$n$$, we can prove that $$R(n, p_i)$$ is a concave function of $$p_i$$, the proof as follows.

Let $$d \left(p_i\right) = \alpha - \beta p_i, d' \left(p_i\right) = -\beta, \hspace{0.5cm} d \left(p_i\right) = 0$$ substitute into Eq. (20), (21), (22), then,

$$\frac{\partial R}{\partial p_k} = \frac{L}{(\eta + 1)n} \left[(1 + \lambda \eta) p_k (\alpha - \beta p_k) + (1 - \lambda) \sum_{i=2}^{n} u_i p_i + \beta an^{-\eta}\right], \hspace{0.5cm} k = 1$$

$$\frac{L}{(\eta + 1)n} \left[\left((1 + \lambda \eta) + (1 - \lambda) u_i v_k\right)(\alpha - 2\beta p_k) + \beta an^{-\eta}w_k - (1 - \lambda) u_i \sum_{j=1}^{k-1} w_j (\alpha - \beta p_j) \right], \hspace{0.5cm} k = 2,3,...,n$$  \hspace{1cm} (32)$$
\[
\frac{\partial^2 R}{\partial p_k^2} = \begin{cases} 
-2\beta L (1+\lambda\eta) & , k=1 \\
\frac{-2\beta L}{(\eta+1)n} ((1+\lambda\eta)+(1-\lambda)u_k v_k) , & k=2,3,\ldots
\end{cases}
\]  \hspace{1cm} (33)

\[
\frac{\partial^2 R}{\partial p_k \partial p_j} = \begin{cases} 
\frac{\beta L (1-\lambda)}{(\eta+1)n} u_k & , k=1, j>k \\
\frac{\beta L (1-\lambda)}{(\eta+1)n} u_k w_j , & 0<j<k, 2\leq k \leq n \\
0 & , j> k, 2\leq k \leq n
\end{cases}
\]  \hspace{1cm} (34)

\[
\frac{\partial^2 R}{\partial p_k^2} + \sum_{j=1, k \neq j}^n \frac{\partial^2 R}{\partial p_k \partial p_j} = \begin{cases} 
\frac{\beta L}{(\eta+1)n} \left(-2(1+\lambda\eta)+(1-\lambda)\sum_{j=2}^n u_j\right) , & k=1 \\
\frac{\beta L}{(\eta+1)n} \left(-2(1+\lambda\eta)-(2(1-\lambda)u_k v_k)+(1-\lambda)u_k \sum_{j=1}^{k-1} w_j\right) , & 2 \leq k \leq n
\end{cases}
\]  \hspace{1cm} (35)

From (35), we can get \( \frac{\partial^2 R}{\partial p_k^2} < -\sum_{j=1, k \neq j}^n \frac{\partial^2 R}{\partial p_k \partial p_j} \), which satisfies the condition of theorem 1, hence, we have completed the proof.

Let \( \frac{\partial R}{\partial p_k} = 0 \), we simplify and organize Eq. (35) as follows.

\[
\begin{align*}
2\beta (1+\lambda\eta) p_i - \beta (1-\lambda) \sum_{j=2}^\infty u_j p_j & = \alpha (1+\lambda\eta) + \beta an^{-\eta} \\
2\beta (1+\lambda\eta) + (1-\lambda)u_k v_k \ p_k - \beta (1-\lambda)u_k \sum_{j=1}^{k-1} w_j p_j & = \alpha (1+\lambda\eta) + (1-\lambda)u_k v_k + \beta an^{-\eta}w_k - \alpha (1-\lambda)u_k \sum_{j=1}^{k-1} w_j , k=2,3,\ldots
\end{align*}
\]  \hspace{1cm} (36)

Given \( p_i (i=1, 2, \ldots, n) \) are an unknown number in Eq. (36). In reality, price adjustment will not be many times in the pre-sale period, this means that the value of \( n \) will not be too large, and the maximum limit of \( n_{\text{max}} \) will be set up by decision makers according to the actual situation. Therefore, different optimal solutions by given different values of \( n \) can be obtained as follows.

a. When \( n=1 \), the optimal number of adjustments \( n^*=1 \), this means that the price will not be changed. According to (18), we can get the max \( R=R(n^*)=R^* \).

b. When \( n<n_{\text{max}} \), let \( n=n+1 \), go to step c; when \( n=\text{max} \), stop and then output result.

c. From (33), we can get the price \( p_i = \{ p_1, p_2, \ldots, p_n \} \).

d. According to the model (24) and Eq. (22), we can get Profit \( R \) and ordering quantity \( Q \).

e. If \( R(n) > R(n^*) \), then \( R^* = R(n) \), \( n^*=n \); or else, return to step b.
VI. NUMERICAL EXAMPLE

Suppose a service product is needed after \( T = 30 \) days and a service organization plans to order and sell this product. The service organization orders this product at a unit cost \( c = 1 \) and sells it through reservation on the following 30 days (the advance selling period). Product demand is shown by the linear function \( d(p) = 10 - p \), this means that \( \alpha=10, \beta=1. \) The situation that customers make cancellations is also considered. The cancellation rate is assumed to decrease with time in accordance with the equation \( r_i = \eta t \). Additionally, the firm is assumed to charge for cancellations with the unit cancellation fee being \( r_i = \lambda p_i = 0.2 p_i \).

a. Assume \( n_{\text{max}} = 7 \), according to the above solving steps, we can obtain the optimal \( n \). The change of profit \( R \) and the initial ordering quantity \( Q \) with \( n \) are shown in Table 1.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>565.78</td>
<td>566.10</td>
<td>566.14</td>
<td>754.78</td>
<td>566.01</td>
<td>565.95</td>
<td>565.88</td>
</tr>
<tr>
<td>( Q )</td>
<td>123.00</td>
<td>123.11</td>
<td>123.97</td>
<td>124.64</td>
<td>125.14</td>
<td>125.44</td>
<td>125.70</td>
</tr>
</tbody>
</table>

Table 1 reveals that the sellers’ revenue \( R \) increases firstly, and then decreases with the increase of \( n \). When \( n = 4 \), the revenue will be the maximum \( R=754.78 \), and the corresponding \( Q=124.64 \). At adjacent time intervals \([0, 7.5), [7.5, 15), [15, 22.5), [22.5, 30)\), the optimal price are 5.70, 5.36, 5.35, 5.36, respectively.

From Table 1, we also can obtain: 1) dynamic pricing is significantly better than static pricing in the pre-sale period; 2) the increase of \( Q \) does not mean that the revenue will increase. That’s to say, the company's total profit and sales volume are not necessarily in a positive correlation; 3) the profit \( R \) and the ordering quantity \( Q \) will be affected by \( n \).

b. Let \( n = 4 \), we will make sensitivity analysis for \( L, c, \lambda, \eta, \alpha, \beta \) as follows.

<table>
<thead>
<tr>
<th>( L )</th>
<th>30</th>
<th>32</th>
<th>34</th>
<th>36</th>
<th>38</th>
<th>40</th>
<th>42</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>754.78</td>
<td>805.10</td>
<td>855.42</td>
<td>905.74</td>
<td>956.06</td>
<td>1006.40</td>
<td>1056.70</td>
</tr>
<tr>
<td>( Q )</td>
<td>124.64</td>
<td>132.95</td>
<td>141.26</td>
<td>149.57</td>
<td>157.88</td>
<td>166.19</td>
<td>174.50</td>
</tr>
</tbody>
</table>

Table 2 reveals that as the pre-sale period extends, the total profit \( R \) and the ordering quantity \( Q \) increase, which is easy to understand.

<table>
<thead>
<tr>
<th>( c )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>754.78</td>
<td>599.44</td>
<td>461.97</td>
<td>342.35</td>
<td>240.60</td>
<td>156.71</td>
<td>90.69</td>
</tr>
<tr>
<td>( Q )</td>
<td>124.64</td>
<td>111.38</td>
<td>98.11</td>
<td>84.86</td>
<td>71.59</td>
<td>58.33</td>
<td>45.07</td>
</tr>
</tbody>
</table>

Table 3 reveals that as \( c \) increases, the total profit \( R \) and the ordering quantity \( Q \) decrease. When other parameters remain unchanged and the \( c \) increases, the sales price will be raised, which will lead to the reduction of the demand for products in the market.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>0.1</th>
<th>0.15</th>
<th>0.2</th>
<th>0.25</th>
<th>0.3</th>
<th>0.35</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>745.96</td>
<td>750.37</td>
<td>754.78</td>
<td>759.21</td>
<td>763.65</td>
<td>768.09</td>
<td>772.53</td>
</tr>
<tr>
<td>( Q )</td>
<td>124.71</td>
<td>124.68</td>
<td>124.64</td>
<td>124.60</td>
<td>124.56</td>
<td>124.52</td>
<td>124.48</td>
</tr>
</tbody>
</table>

Table 4 reveals that with the increase of \( \lambda \), the total profit \( R \) increases and the ordering quantity \( Q \) decreases.

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
<th>0.10</th>
<th>0.11</th>
<th>0.12</th>
<th>0.13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>763.25</td>
<td>760.40</td>
<td>757.58</td>
<td>754.78</td>
<td>752.01</td>
<td>749.26</td>
<td>746.53</td>
</tr>
<tr>
<td>( Q )</td>
<td>132.89</td>
<td>130.08</td>
<td>127.33</td>
<td>124.64</td>
<td>122.02</td>
<td>119.46</td>
<td>116.95</td>
</tr>
</tbody>
</table>

Table 5 reveals that the increase of \( \eta \) will lead to the decline of the total profit \( R \) and the ordering quantity \( Q \).
Table 6 reveals that a larger market α means greater total profit \( R \) and the ordering quantity \( Q \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>336.15</td>
<td>457.14</td>
<td>596.68</td>
<td>754.78</td>
<td>931.45</td>
<td>1126.70</td>
<td>1340.5</td>
</tr>
<tr>
<td>( Q )</td>
<td>83.27</td>
<td>97.06</td>
<td>110.85</td>
<td>124.64</td>
<td>138.43</td>
<td>152.22</td>
<td>166.01</td>
</tr>
</tbody>
</table>

Table 7 reveals that with the increasing price-sensitive \( \beta \), the ordering quantity \( Q \) and the total profit \( R \) decrease. To be specific, the increase of the price-sensitive \( \beta \) will cause the rise of the price, which will lead to the decline of the market demand, then the reduction of the ordering quantity \( Q \) and the total profit \( R \).

VII. Conclusions

This paper has discussed dynamic pricing for service products to maximize the revenue of sellers. A mathematical model is set up to find an optimal solution of the objective function. Assuming that the relationship between demand and price is linear, for any given \( n \), the profits function \( R(n, p_i) \) is a concave function of price \( p_i (i = 1, 2, \cdots, n) \), and at last, an algorithm for the optimal solution has been given.

Moreover, the initial ordering quantity \( Q \) and the total profit \( R \) have the following characteristics: 1) the optimal number of price adjustments in the pre-sale period is \( n=4 \); with the increase of \( n \), the sellers’ revenue \( R \) first increases, then decreases while the initial ordering quantity \( Q \) keeps increasing. 2) the total profit \( R \) and the ordering quantity \( Q \) will increase with the extension of pre-sale period \( L \). 3) the total profit \( R \) and the ordering quantity \( Q \) will decrease with the increasing unit purchasing cost \( c \). 4) with the increase of unit penalty cost for cancellation during time \( \lambda \), the total profit \( R \) will increase while the ordering quantity \( Q \) will decrease. 5) more times of price adjustments in the pre-sale period will lead to the decline of the total profit \( R \) and the ordering quantity \( Q \). 6) The larger the market \( \alpha \) is, the greater the total profit \( R \) and the ordering quantity \( Q \) will be. 7) the ordering quantity \( Q \) and the total profit \( R \) will decrease with the increasing price-sensitive \( \beta \).

The future work should be carried out as follows:

1) Market demand is not only influenced by price, but also influenced by other factors such as time, etc.; therefore, it’s limited to just consider the impact of the price.

2) In most cases, demand is stochastic. Therefore, it seems necessary to reformulate the model as a dynamic programming model and requires us to develop another approach to deal with the new model.

3) Customers’ strategic behaviour of dynamic pricing should be considered in the model, which will be of more practical significance.

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