New Homoclinic Solution for Davey-Stewartson Equation with Periodic Boundary

Zhengde Dai¹, Xiping Zeng², Murong Jiang³, Chuanjian Wang⁴

School of Mathematics and Statistics, Yunnan University, Kunming, 650091, China
Department of Information and Computing Science, Guangxi Institute of Technology, Liuzhou, 545006, China
School of Information Science and Engineering, Yunnan University, Kunming, 650091, China
School of Science, Kunming University of Science and Technology, Kunming, 650093, China

¹zhddai2004@yahoo.com.cn; ²zengxiping@163.com; ³jiangmr@ynu.edu.cn; ⁴wcj20082002@yahoo.com.cn

Abstract- A new kind of the homoclinic solution with oscillatory structure for Davey-Stewartson (DSI and DSII) equation with periodic boundary condition is constructed by using the Hirota’s bilinear form and extended homoclinic test method, respectively. The mechanical feature of the solution is also investigated. Result shows the variety of the structure for the homoclinic solution of one integrable system with periodic boundary condition.

Keywords- Homoclinic Solution; Breather; Davey-Stewartson Equation; Periodic Boundary

I. INTRODUCTION

Davey-Stewartson (DS) equation is written as [1]:

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathrm{i}u_t &= -u_{xx} - \frac{4}{\alpha_0} u_{yy} - \frac{2}{\alpha_0^2} |u|^2 u - \frac{1}{\alpha_0^2} uv \\
v_{yy} - \alpha_0^2 v_{xx} - 2\alpha_0^2 |u|^2 &= 0
\end{array} \right.
\]

(1.1)

where \( u : R^+ \times R^+ \times R^+ \to C, v : R^+ \times R^+ \times R^+ \to R \). \( \alpha \) and \( \alpha_0 \) are constants. DS equation was derived by Davey et al. to model the evolution of a three-dimensional disturbance in the nonlinear regime of plane Poiseuille flow. The function \( u(x, y, t) \) stands for the complex amplitude, and \( v(x, y, t) \) describes the perturbation of the real velocity. DS equation is called the DSI as \( \varepsilon = 1, \alpha_0 = \pm 1 \) and DSII as \( \varepsilon = 1, \alpha_0 = \mp i \).

There are known results due to local well-posed, global existence and blow-up of some solutions, exact periodic soliton solutions, solitoff and dromion solutions [2-11]. Recently, homoclinic and heteroclinic tube solutions were obtained [12-15].

It is well known that the existence of homoclinic and heteroclinic orbits solutions is very important for studying the spatiotemporal chaos of partial differential equation. Many methods were developed for proving the existence of homoclinic orbits of perturbed soliton equation. As we know that the homoclinic solution is non-wave type solution, it is generally obtained using “Homoclinic test method” and can be expressed by function \( F(\cos(p_x x + p_y y - \alpha t), \cosh(p_x x + p_y y - \beta t + \delta)) \). The two-wave kind of homoclinic solution satisfies periodic boundary condition and asymptotically tends to a fixed cycle as time tends to infinity. We also exhibit locally structure of these solutions, respectively.

Consider DSI equation

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathrm{i}u_t + u_{xx} + u_{yy} &= -2 |u|^2 u - 2uv \\
v_{xx} - v_{yy} &= -2(|u|^2)_{xx}
\end{array} \right.
\]

(1.2)

with periodic boundary condition

\( u(x, y, t) = u(x + l_1 + y + l_2, t); \quad v(x, y, t) = v(x + l_1 + y + l_2, t) \) (1.3)

and DSII equation

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathrm{i}U_t + U_{xx} - U_{yy} &= 2 |U|^2 U + 2UV \\
V_{xx} + V_{yy} &= -2(|U|^2)_{xx}
\end{array} \right.
\]

(1.4)

with periodic boundary condition

\( U(x, y, t) = U(x + l_1, y + l_2, t); \quad V(x, y, t) = V(x + l_1, y + l_2, t) \) (1.5)

In this work, we analyse linear stability in neighbourhood of fixed cycle and then use the extended homoclinic test approach [16-18] to construct a new kind of the homoclinic solution different form homoclinic tube solution for DS equation [12, 13], some mechanical features are investigated and global structure of the solutions is exhibited.

II. HOMOCLINIC BREATHER SOLUTION FOR DSI EQUATION

It is easy to see that \( \phi_{ex}(ia^2 t, 0) \) is a fixed cycle of DSI equation [13]. We investigate the linear stability of fixed cycle by considering a small perturbation of the form

\[
\begin{align*}
\phi &= \phi_{ex}(ia^2 t, 0) + q_{ex}(x, y, t) \\
\phi &= \phi_{ex}(ia^2 t, 0) + q_{ex}(x, y, t)
\end{align*}
\]

(2.1)

where \( q_{ex}(x, y, t) \ll 1 \). Substituting Eq. (2.1) into Eq. (1.1), we get the linearized equation as

\[
\begin{align*}
\dot{q}_{ex} + q_{ex} + q_{ey} &= -a^2 q_{ex} + \phi \\
\phi_{ex} - \phi_{ey} &= 2a^2 q_{ex} + 2a^2 q_{ey}
\end{align*}
\]

(2.2)
where \( \ast \) denote the conjugation. Assume that \( q_e \) and \( \phi_e \) have the following forms
\[
\begin{align*}
q_e &= Ae^{i(\mu_n x + \pi_n y)} + Be^{-i(\mu_n x + \pi_n y)} + \sigma_n, \\
\phi_e &= C(e^{i(\mu_n x + \pi_n y)} + e^{-i(\mu_n x + \pi_n y)})
\end{align*}
\] (2.3)
where \( A, B \) are complex constants and \( C \) is real,
\( \mu_n = \frac{2\pi}{\ell_n} \), \( \pi_n = p_n + \frac{2\pi}{\ell_n} \) and \( \sigma_n \) is the growth rate of the \( n \)th mode.

Substitution of Eq. (2.3) into Eq. (2.2) leads to
\[
\begin{align*}
(i\pi_n - \mu_n^2 + \pi_n^2 + a^2)A &= -a^2 B + c \\
(i\pi_n - \mu_n^2 + \pi_n^2 + a^2)B &= -a^2 A + C \\
-c\mu_n^2 + C\pi_n^2 &= 2a^2 A\mu_n^2 + 2a^2 B\pi_n^2 \\
C\mu_n^2 + C\pi_n^2 &= 2a^2 B\mu_n^2 + 2a^2 A\pi_n^2
\end{align*}
\] (2.4)
Solving Eq. (2.4), we obtain
\[
\begin{align*}
\sigma_n^2 &= (\mu_n^2 + \pi_n^2)(2a^2 - \mu_n^2 - \pi_n^2) \\
\sigma_n^2 + \pi_n^2 &= 2a^2
\end{align*}
\] (2.5)
This shows that the fixed cycle is hyperbolic provided.
\[
n^2 < \frac{2a^2}{p_i^2 + p_i^2}
\] (2.7)
Thus the number of unstable modes which determines the complexity of the homoclinic structure is given by the following the largest integer \( N \) with
\[
0 < N < \frac{\sqrt{2}|a|}{\sqrt{p_i^2 + p_i^2}}
\] (2.8)
Now, by using extended homoclinic test approach\cite{16,17}, we construct the homoclinic breather solution of DSI equation.

Make the transformation \( u = \frac{\sqrt{2}}{2}\exp(i\alpha^2)Q, v = -\frac{\varphi}{2} \) and substitute it into Eq. (1.1), we can get
\[
\begin{align*}
[iQ_x + Q_{xx} + Q_{yy} &= -a^2(|Q|^2 - 1)Q + Q\phi \\
\varphi_{xx} - \varphi_{yy} &= 2a^2(|Q|^2)\varphi
\end{align*}
\] (2.9)
where \( Q = Q(x, y, t) \) is a complex function, and \( \varphi \) is a real.

By the dependent variable transformation
\[
Q = \frac{G}{F}, \quad \varphi = -4(\ln F)_{xx}
\] (2.10)
where \( G \) is a complex and \( F \) a real. Then, Eq. (2.9) can be converted into the form
\[
\begin{align*}
[iG_F - IF_G + G_{xx}F - 2G_F F_x + G_{yy}F &- 2GF_x + GF_{yy} - (a^2 + B)GF = 0, \\
2(F_{yy}F - F_{xx}F - F_x F_y - F_{yy}) - BF^3 - a^2GG^2 &= 0
\end{align*}
\] (2.11)
where \( B \) is an integral constant and \( \ast \) denotes the complex conjugation.

By means of the extended homoclinic test approach\cite{16,17}, we take the test function as follows:
\[
G = e^{p(x+y) \cos(p(x+y))} + a_1 \sin(p(x+y)) + a_2 e^{p(x+y)}
\] (2.12)
\[
F = e^{p(x+y) \cos(p(x+y))} + a_1 \sin(p(x+y)) + a_2 e^{p(x+y)}
\] (2.13)
where all of \( a_1, a_2, p, \alpha, \alpha_1 \) are real, and \( a_1, a_2 \) are complex. Substituting Eq. (2.12) into Eq. (2.11), equating the coefficients of all powers of \( e^{p(x+y) \cos(p(x+y))} \), \( e^{p(x+y)} \sin(p(x+y)) \) and \( e^{p(x+y)} \) \((j = -1, 0, 1)\) to zero, we can obtain a set of algebraic equations for \( p, p, \alpha, \alpha_1 \) \( k = 1, 2, 3, 4 \) with
\[
B = -a^2
\]
\[
\begin{align*}
4pp_1 - ip\alpha a_1 + (4pp_1 + ip\alpha a_2) &= 0 \\
(-ip\alpha + 4pp_1 + ip\alpha a_1) + (4pp_1 + ip\alpha) a_2 &= 0 \\
(ip\alpha + 4pp_1 - 5p_1) a_1 + (4p_1 - 5p_1) a_2 &= 0 \\
(-10a_1 - p_1 + (5p^2 + 2ip\alpha) a_1 + (5p^2 - 2ip\alpha) a_2 &= 0 \\
(a^2(a_2 - a_1)) &= 0 \\
(a_2^2 - a_1^2) &= 0 \\
(2a^2 - \frac{1}{4}p_1^2 + 6p_1) a_1 - a^2(a_2 + a_1) &= 0 \\
(2a^2 - 6p_1^2) a_2 - 6a_1^2 - a^2(a_2 + a_1) &= 0
\end{align*}
\]
Solving these equations, we obtain the relations between the parameters as
\[
b = -a^2, \quad p_1^2 = \frac{21p_1^2}{40}, \quad p_2^2 = \frac{320a^2 - 39a^2}{624}
\] (2.13)
\[
a_1 = \frac{(i\alpha + 4p)}{i\alpha - 4p}, \quad a_2 = \frac{(i\alpha + 4p) a_1}{i\alpha - 4p} = \frac{42(a^2 - 80p_1^2) a_1}{21(a^2 + 16p_1^2)}
\]
From \( p_1^2 \geq 0 \) and \( a_1^2 \geq 0 \) in Eq. (2.13), we have \( 800a^2 \leq a_1^2 \leq \frac{320a^2}{59} \). Substituting Eq. (2.13) into Eq. (2.12) and then Eq. (2.10), taking \( a_1 > 0 \), we obtain the solution for DSI equation as
\[
\begin{align*}
u &= \frac{a}{\sqrt{2}} e^{i(\alpha, x, y)} \\
\psi &= \frac{2H(\zeta, \eta, \gamma)}{a_1 \cos(\zeta + \gamma)} + \frac{2\sin(\zeta + \gamma)}{\sqrt{2} \cos(\zeta + \gamma)}
\end{align*}
\] (2.14)
where
\[
H(\zeta, \eta, \gamma) = 2a_1 \sqrt{a_1 (p_1^2 - p_2^2)} \cos(\zeta + \gamma) \\
+ 4a_1 p_1 \sqrt{a_1} \sin(\zeta + \gamma) + 4a_1 p_2 - a_1 p_2^2
\]
and \( \zeta = p(x + \gamma + at), \quad \eta = p(x + 2y - at), \quad \gamma = \ln \sqrt{a_1} \) and \( e^{i\theta} = \frac{(i\alpha + 4p)}{i\alpha - 4p}, \quad p, p, \alpha, a_1 \) are given by Eq. (2.13). Note that if \((a(x, y, t), v(x, y, t))\) is the solution of DSI equation, then \((a(x, y, t), v(\gamma - y, t))\) is the solution as well. So, we also obtain the solution of DSI equation as
such that all wave numbers is the solution as well.

\[ (3.1) \]

\[ (3.2) \]

\[ (3.3) \]

\[ (3.4) \]

\[ (3.5) \]

III. HOMOCLINIC BREATHER SOLUTION FOR DSII EQUATION

As we know that \((a \exp(-2i |a|^2 t), 0)\) is a hyperbolic fixed cycle of DSII equation when the period of \(y\) is larger than the period of \(x\) \[12\]. Similar to the argument in \[12\], we can analyse the linear stability of fixed cycle \((a \exp(-2i |a|^2 t), 0)\). By similar process of dealing with of Eq. (1.1), we take the transformation

\[ U = \frac{G}{F}, \quad V = A - 2(\ln F)_{,x} \]

Eq. (1.3) can be converted into the bilinear form

\[ \left\{ \begin{array}{l}
(iD_x + D_y - D_x^2)G \cdot F = (\lambda + 2A)G \cdot F \\
(D_x^2 + D_y + \lambda)F \cdot F = 2GG^* 
\end{array} \right. \]

where \(A\) is a constant, \(G\) is a complex function, \(F\) is a real. Now, we take the following ansatz:

\[ G = ae^{i2\omega t} \left[ e^{-\frac{p_1}{2}(x^2 + y^2 - a^2)} + b_1 \cos(p_2(x - \sqrt{2}y + at)) \right] + b_2 e^{i\alpha_2}(x^2 + y^2 - a^2) \]

\[ F = e^{-\frac{p_1}{2}(x^2 + y^2 - a^2)} + b_1 \cos(p_2(x - \sqrt{2}y + at)) \]

\[ + b_2 e^{i\alpha_2}(x^2 + y^2 - a^2) \]

where \(a, p_1, p_2, \alpha_2, b_1, b_2\) are real, \(b_1, b_2\) are complex. Substituting Eq. (3.3) into Eq. (3.2), we get

\[ \lambda = 2a^2, \quad A = -a^2, \quad p_1 = \frac{64a^2 + 3(1 - 2\sqrt{2}a\xi)}{96(2\sqrt{2} - 1)} \]

\[ p_2 = \frac{(4\sqrt{2} - 1)p_1}{i\alpha - 4\sqrt{2}p_1} \]

\[ b_2 = \left( \frac{i\alpha + 4\sqrt{2}p_1}{i\alpha - 4\sqrt{2}p_1} \right)^3 b_1 \]

where \(\alpha^2 < \frac{64a^2}{32 - 2\xi}\). Substituting Eq. (3.4) into Eq. (3.1), we obtain the following solution for DSII equation:

\[ \left\{ \begin{array}{l}
U = ae^{i\theta - 2i|\alpha| t} \left[ \frac{2\cosh(p_1(x + \sqrt{2}y - at)) + b_1 \cos(R_1(x, y, t))}{2\cosh(p_1(x + \sqrt{2}y - at)) + b_1 \cos(R_1(x, y, t))} \right] \\
V = \frac{2M}{b_2(2\cosh(p_1(x + \sqrt{2}y - at)) + b_1 \cos(R_1(x, y, t)))} 
\end{array} \right. \]

where

\[ R_1(x, y, t) = p_1(\sqrt{2}x + y - at) + \ln \sqrt{b_1} \]

\[ R_2(x, y, t) = p_1(x - \sqrt{2}y + at) \]

and

\[ M = 8b_2 - (4\sqrt{2} - 1)b_1^2 p_1^2 - 2p_1^2 b_2 \sqrt{b_1} \left((3 - 4\sqrt{2})\cosh(R_1(x, y, t))\cos(R_1(x, y, t)) \right) - 32\sqrt{2} - 8 \sin(R_1(x, y, t)) \sin(R_1(x, y, t)) \]

\[ p_1, p_2, \alpha, b_1, b_2, b_3, b_4 \] are given by Eq. (3.4) and \(e^{i\theta} = \frac{ia^2 + 4\sqrt{2}p_1}{a^2 - 4\sqrt{2}p_1} \).

Note that if \((U(x, y, t), V(x, y, t))\) is the solution of DSII equation, then \((U(x, -y, t), V(x, -y, t))\) is the solution as well. So, we obtain the solution of DSII equation.
in DSII and the homoclinic breather variation in plane, then we obtain a new kind of homoclinic solutions of two-wave extended homoclinic test method to DSI and DSII equations, and the oscillation of two-wave for DSII is stronger (Ref. Fig. 3).

\[
\begin{align*}
U_1 &= a e^{i(\theta - 2\pi t/3)} \frac{2 \cosh (R_1(x, y, t) + i \theta) + \frac{1}{\sqrt{b_1}} \cos (R_1(x, y, t))}{2 \cosh (R_1(x, y, t)) + \frac{1}{\sqrt{b_1}} \cos (R_1(x, y, t))} \\
V_1 &= -\frac{2M_1}{b_1} (2 \cosh (R_1(x, y, t)) + \frac{1}{\sqrt{b_1}} \cos (R_1(x, y, t)))^2
\end{align*}
\]  

(3.6)

where

\[
\begin{align*}
R_1(x, y, t) &= p_1 \sqrt{2x - y - \alpha t} + \ln \sqrt{R_1} \\
R_1(x, y, t) &= p_1 \sqrt{x + \sqrt{2} y + \alpha t}
\end{align*}
\]

and

\[
M_1 = (8b_1 - (4\sqrt{2} - 1)b_1^2) p_1^2 + 2p_1^2 b_1 \sqrt{R_1} \left[ (3 - 4\sqrt{2}) \cos (R_1(x, y, t)) \cosh (R_1(x, y, t)) - \sqrt{32} - 8 \sin (R_1(x, y, t)) \sinh (R_1(x, y, t)) \right]
\]

The solution given by Eq. (3.6) is a homoclinic breather solution different from the homoclinic tubes solution obtained in [12]. It satisfies the periodic boundary condition. In fact, if we take \( I_1 = \frac{2\pi}{3} p_1 \), then

\[
\begin{align*}
U_1(x, y, t) &= U_1(x + I_1, y + I_1, t) \\
V_1(x, y, t) &= V_1(x + I_1, y + I_1, t)
\end{align*}
\]

It is obvious that Eq. (3.6) is a two-wave kind of homoclinic solution, which has similar structure to Eq. (2.15). But the periodic boundary for DSI is different from DSII. Specially, the oscillation of two-wave for DSII is stronger (Ref. Fig. 3 and 4).

![Fig. 3 Behaviour of \( U_1 \) in DSI and the homoclinic breather variation in \( x - U_1 \) Plane](image_url)

![Fig. 4 Behaviour of \( V_1 \) in DSI and the homoclinic breather variation in \( x - V_1 \) plane](image_url)

IV. CONCLUSIONS

Based on the Hirota bilinear form, by applying the extended homoclinic test method to DSI and DSII equations, we obtain a new kind of homoclinic solutions of two-wave type with locally oscillatory structure. We also investigate and exhibit the different homoclinic structures of solutions. These results show the complexity and variety of dynamical behavior for DS system. Following these ideas in this work, the problem needed to be further studied is whether the other types of nonlinear evolutions have this kind of homoclinic solutions or not.

ACKNOWLEDGMENT

This work was supported by Chinese Natural Science Foundation Grant No. 11061028 and No. 11161055.

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Murong Jiang was born in Hunan, China, on June, 1963. She received the B.S. degree in mathematics in 1984 and the M.S. degree in mathematics in 1989 from Yunnan University, China. She received the Ph.D. degree in mathematics in 1999 from the Institute of Applied Physics and Computational Mathematics of Beijing, China. Now she is the professor in the Department of Computer Science and Engineering, Yunnan University. Her research interests include partial differential equation and its application in parallel image processing.

Chuanjian Wang was born in Yunnan, China, on September, 1984. He received the B.S. degree in mathematics in 2007 from Chuxiong Normal University and the M.S. degree in mathematics in 2010 from Yunnan University, China. Now he is the teaching assistant in the School of Science, Kunming University of Science and Technology. His research interests include partial differential equation and integrable system.