New Exact Solutions for the VGKdV-mKdV Equation with Nonlinear Terms of Any Order

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Abstract-In this paper, some new exact solutions for the variable-coefficient generalized KdV-mKdV equation (VGKdV-mKdV) with nonlinear terms of any order are obtained by using the generalized Jacobi elliptic functions expansion method with computerized symbolic computation, some of these solutions are degenerated to soliton-like solutions and trigonometric function solutions in the limit cases, which shows that the applied method is more powerful and will be used in further works to establish more entirely new exact solutions for other kinds of nonlinear partial differential equations with nonlinear terms of any order arising in mathematical physics.

Keywords- Generalized Jacobi Elliptic Functions Expansion Method; Generalized KdV–mKdV Equation; Exact Solutions; Soltion-Like Solutions; Jacobi Elliptic Wave-Like Solutions

I. INTRODUCTION

Looking for exact solutions to nonlinear evolution equations (NEEs) has long been a major concern for both mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields, such as solitons and propagation with a finite speed, and thus they may give more insight into the physical aspects of the problems. Up to now, many effective methods have been presented, such as inverse scattering transformation [1], Hirota bilinear method [2], homogeneous balance method [3], B¨acklund transformation [4], Darboux transformation [5], the extended tanh-function method [6], the extended F-expansion method [7], projective Riccati equations method [8], the Jacobi elliptic function expansion method [9] and so on [10].

The main goal of this paper is to find the new and more general exact solutions of the VGKdV-mKdV equation by using the generalized Jacobi elliptic functions expansion method [11, 12] proposed recently. The character feature of our method is that, without much extra effort, we can get series of exact solutions using a uniform way. Another advantage of our method is that it also applies to general nonlinear differential equations with nonlinear terms of any order.

This paper is arranged as follows. In Section 2, we briefly describe the generalized Jacobi elliptic function expansion method. In Section 3, several families of solutions to the VGKdV-mKdV equation are obtained. In Section 4, some conclusions are given.

II. SUMMARY OF THE GENERALIZED JACOBI ELLIPTIC FUNCTIONS EXPANSION METHOD

For a given partial differential equation, say, in two variables and

\[ P(u, u_{xx}, u_{x}, u_{xt}, u_{t}, u_{tt}, \cdots) = 0 \] (1)

We seek the following formal solutions of the given system:

\[ u(\xi) = \sum_{j=0}^{N} A_{j} F_{j}(\xi) + \sum_{i, j=1, j \neq i \leq N}^{N} [B_{ij} F_{i}(\xi) F_{j}(\xi)] + C_{i} F_{i}(\xi) G_{i}(\xi) + D_{i} F_{i}(\xi) H_{i}(\xi) \] (2)

where \( A_{0}, A_{1}, B_{1}, C_{1}, D_{1}, (i = 1, 2, \cdots, n) \) are time-depend-ent functions to be determined later. \( \xi = \xi(x, t) \) are arbitrary functions with the variables \( x \) and \( t \). The parameter \( N \) can be determined by balancing the highest order derivative terms with the nonlinear terms in Eq.(2). And \( E(\xi), F(\xi), G(\xi), H(\xi) \) are an arbitrary array of the four functions \( e = e(\xi), f = f(\xi), g = g(\xi) \) and \( h = h(\xi) \), the selection obey the principle which makes the calculation simpler. Here we ansatz

\[
\begin{aligned}
\varepsilon &= \frac{1}{p+q sn_{n}z + r cn_{n}z + l dn_{n}z}, \\
f &= \frac{sn_{n}z}{p+q sn_{n}z + r cn_{n}z + l dn_{n}z}, \\
g &= \frac{cn_{n}z}{p+q sn_{n}z + r cn_{n}z + l dn_{n}z}, \\
h &= \frac{dn_{n}z}{p+q sn_{n}z + r cn_{n}z + l dn_{n}z}
\end{aligned}
\] (3)

where \( p, q, r, l \) are arbitrary constants, the four function \( e, f, g, h \) satisfy the restricted relations(4) and (5a-5d) mentioned in [11, 12].

Substituting (2) along with (4) (5a-5d) into Eq.(1) separately yields four families of polynomial equations for \( E(\xi), F(\xi), G(\xi), H(\xi) \). Setting the coefficients of \( F_{i}(\xi) E_{i}(\xi) G_{i}(\xi) H_{i}(\xi) \) \( (j_{1}, j_{2}, j_{3}, j_{4}) = (0, 1, 2, \cdots) \) to zero yields a set of over-determined differential equations (ODEs) in \( A_{0}, A_{1}, B_{1}, C_{1}, D_{1}, (i = 1, 2, \cdots, n) \) and \( \xi(x, t) \), solving the ODEs by Mathematica and Wu elimination, we can obtain many exact solutions of Eq.(1) according to (2) and (3) .

III. EXACT SOLUTIONS OF THE GENERALIZED KDV–MKDV EQUATION

The variable-coefficient generalized KdV–mKdV equation with nonlinear terms of any order
If one takes any class of physically important equations. arbitrary positive integer, \( u = u(x, t), \gamma(t), \alpha(t), M(t), \beta(t) \) and the forced term \( R(t) \) are arbitrary functions of \( x, t \). Eq. (6) includes a class of physically important equations.

In fact, if one takes \( \gamma(t) = M(t) = 0, n = 1 \), Eq. (4) represents VKdV equation [10-15]

\[
u_t + \alpha(t)\nu u_x + \beta(t)u_{xxx} = R(t)
\]

If one takes \( \gamma(t) = 0, M(t) \neq 0, n = 1 \), Eq. (4) becomes VCKdV equation [16-17]

\[
u_t + \alpha(t)\nu u_x + M(t)u^2u_x + \beta(t)u_{xxx} = R(t)
\]

If one takes \( \gamma(t) = M(t) = R(t) = 0, \beta(t) = 1 \), Eq. (4) turns to GKdV-mKdV equation [20-24]

\[
u_t + \alpha(t)\nu^n u_x + u_{xxx} = 0
\]

In the follows, we construct exact solutions of Eq. (4).

Using transformation \( u = v^n \) yield

\[
\begin{align*}
\nu^2 v_x + \gamma v^2 v_x + \frac{1}{2}\nu^3 v_x + \beta(1 - n)(1 - 2n) \left( \frac{n^2}{n^2} \right) v_x^2 \\
+ 3(1 - n) v^2 v_x + \nu^3 v_{xxx} &= n v^{1 - \frac{1}{n}} R(t)
\end{align*}
\]

Where

\[
u = u(x, t), v = v(x, t), \gamma = \gamma(t), \alpha = \alpha(t), M = M(t), \beta = \beta(t).
\]

Making the gauge transformation

\[
\xi = k(t)x + \omega(t)
\]

We have

\[
\begin{align*}
\left( \alpha_1 + k \right) \nu^2 v_x + k \gamma \nu^2 v_x + k \alpha \nu^2 v_x + k M \nu^4 v_x \\
+ k^n \beta(1 - n)(1 - 2n) \left( \frac{n^2}{n^2} \right) v_x^2 + 3(1 - n) v^2 v_x &+ \nu^3 v_{xxx} = n v^{1 - \frac{1}{n}} R(t)
\end{align*}
\]

where

\[
k = k(t), \omega = \omega(t), \alpha_1 = \frac{d\omega}{dt}, k_1 = \frac{dk}{dt}, v_x = \frac{dv}{d\xi}.
\]

\[
v_{\xi} = \frac{d^2v}{d\xi^2}, v_{\xi\xi} = \frac{d^3v}{d\xi^3}, k(t), \omega(t) \text{ are functions of } t
\]

to be determined later.

A. The Jacobi Elliptic Function Solutions to Eq. (5)

By balancing the highest-order linear term \( u^n \) and the nonlinear \( uu' \) in (5), we obtain \( N = 2 \), thus we assume that (5) have the following solutions:

\[
\begin{align*}
\alpha &= c_0 + c_1 e + c_2 f + c_3 g + c_4 h + d_1 e^2 + d_2 f^2 + d_3 g^2 \\
&+ d_4 h^2 + d_5 fg + d_6 fh + d_7 gh + d_8 ef + d_9 eg + d_{10} eh
\end{align*}
\]

Where

\[
u = u(\xi), e = e(\xi), f = f(\xi), g = g(\xi), h = h(\xi),
\]

and (5a-5d) separately along with (12) into (5) and setting the coefficients of \( F^l(\xi)E^m(\xi)^n G(\xi)^h H(\xi)^k \) (\( i = 0, 1, 2, \cdots \)) \( (j = 0, 1, \cdots, 10) \) to zero yield an ODEs with respect to the unknowns \( c_i(i = 0, \cdots, 4), d_j(j = 1, \cdots, 10), \alpha, k, p, q, r, l, m \). After solving the ODEs by Mathematica and Wu elimination we could determine the following solutions:

Case 1

\[
r = l = 1, q = \pm 1, \beta(t) = C\alpha(t), c_0 = \int R(t) dt,
\]

\[
d_1 = \frac{3Ck^2(m^2 - 2)^2}{(m^2 - 1)}, d_4 = \frac{3Ck^2m^4}{m^2 - 1},
\]

\[
c_3 = -6k^2C(m^2 - 2), c_4 = 6Ck^2m^2,
\]

\[
\omega(t) = \int \left[-k\alpha(t)\right] R(t) dt - k^3 C(2m^2 + 5)\alpha(t) dt
\]

Case 2

\[
\beta(t) = C\alpha(t), c_0 = \int R(t) dt,
\]

\[
d_1 = \frac{3k^2 p^2 m^4 C}{(m^2 - 1)}, l = \pm \frac{p}{\sqrt{1 - m^2}},
\]

\[
\omega(t) = \int \left[-k\alpha(t)\right] R(t) dt - 2Ck^3(m^2 - 2)\alpha(t) dt
\]

Case 3

\[
p^2 = 1, q^2 = 1, r = \pm 1, \beta(t) = C\alpha(t),
\]

\[
c_3 = \pm 12k^2 C, d_1 = -12Ck^2, c_0 = \int R(t) dt,
\]

\[
\omega(t) = \int \left[-k\alpha(t)\right] R(t) dt + Ck^3(4m^2 - 5)\alpha(t) dt
\]

Case 4

\[
q = r = 0, \beta(t) = C\alpha(t),
\]

\[
d_1 = 12Ck^2m^2p^2, c_0 = \int R(t) dt,
\]

\[
\omega(t) = \int \left[-k\alpha(t)\right] R(t) dt + 4Ck^3(1 - 2m^2)\alpha(t) dt
\]
where \( C \neq 0, k \neq 0 \) are arbitrary constants in Case 1-Case 4. The other coefficients \( c_i, d_j (i = 1, \ldots, 4; \quad j = 1, \ldots, 10) \) don’t mention in all above cases are zero. Therefore from (3), (10), (12) and Cases 1–4, we obtain the Jacobi elliptic wave-like solutions to Eq. (5):

\[
\begin{align*}
\xi_1 &= kx + \int \left[-k\alpha(t) \int R(t) dt - k^3 C(2m^2 + 5) \alpha(t) \right] dt \\
\xi_2 &= kx + \int \left[-k\alpha(t) \int R(t) dt - 2k^3 (m^2 - 2) \alpha(t) \right] dt \\
\xi_3 &= kx + \int \left[-k\alpha(t) \int R(t) dt + Ck^3 (4m^2 - 5) \alpha(t) \right] dt \\
\xi_4 &= kx + \int \left[-k\alpha(t) \int R(t) dt + 4Ck^3 (1 - 2m^2) \alpha(t) \right] dt
\end{align*}
\]

Remark 1: If we let \( \int R(t) dt = \text{const.} \), \( k \to p, C \to 1, m \to k \) or 1, \( u_{1,4} \) is equivalent to the Solutions (23, 25, 26) and the famous bell-shape soliton Solution (48) given in [18]. If we let \( \int R(t) dt = 0, k \to p, m \to k, C \to 1 \), \( u_{1,2} \) is equivalent to the Solution (47) given in [18]. Solutions \( u_{1,4}(\xi) (i = 1, 3, 4) \) are degenerated to soliton-like solutions when the modulus \( m \to 1 \), and solutions \( u_{1,4}(\xi) (i = 1, 3) \) are degenerated to trigonometric functions solutions when the modulus \( m \to 0 \).

The typical structure of new Jacobi elliptic wave-like solution \( u_{1,2}, u_{1,4} \) is shown in Fig.1 and Fig.2 with \( R(t) = 0 \). Here \( u_{1,4} \) provides us a Jacobi cnoidal wave solution and the famous bell-shaped soliton solution when \( m \to 1 \) which covers the large majority of physically interesting solitary waves.

\[
\begin{align*}
\text{Fig. 1 Simulation of } U_{1,2} \\
\text{Fig. 2 Simulation of } U_{1,4}
\end{align*}
\]

when \( \alpha(t) = k = C = 1, \epsilon = -1, m = 0.64 \) and \( t = 0 \).

B. The Jacobi Elliptic Function Solutions to Eq. (6)

By balancing the highest-order linear term \( u^m \) and the nonlinear \( u^2 u' \) in (6), we obtain \( N = 1 \), thus we assume that (6) have the following solutions:

\[
u = c_0(t) + c_1(t)e + c_2(t)f + c_3(t)g + c_4(t)h \tag{13}
\]

with the similar process, we obtain the following solutions:

**Case 1**

\[
\begin{align*}
c_0(t) &= \int R(t) dt, \quad c_1(t) = c_2(t) = c_3(t) = 0, \\
c_4(t) &= \pm k\frac{\sqrt{3C((p^2 - r^2)^2(m^2 - 1) - 4p^2r^2)}}{2M(t)}
\end{align*}
\]

\[
\begin{align*}
of(t) &= \int \left[ \frac{k\alpha(t)^2}{4M(t)} - 3CM(t) \right] dt, \\
\int R(t) dt &= -\frac{\alpha(t)}{2M(t)} + kr((p^2 + r^2) + m^2(p^2 - r^2)) \\
&\times \sqrt{\frac{3C}{(m^2(p^2 - r^2)^2 - (p^2 + r^2)^2)}}
\end{align*}
\]

**Case 2**

\[
\begin{align*}
p = 1, q = 0, r \neq 0, \beta(t) &= 2CM(t), \quad k = k, \\
c_0(t) &= \int R(t) dt, \quad c_1(t) = c_2(t) = c_3(t) = 0, \\
c_4(t) &= \pm k\sqrt{12C(r^2 - 1)(m^2(r^2 - 1) - r^2)}, \\
of(t) &= \int \left[ \frac{k\alpha(t)^2}{4M(t)} - k^2CM(t) \right] dt \\
&\times \sqrt{3C} \left( (1 - r^2)(r^2 + m^2(1 - r^2)) \right)
\end{align*}
\]

\[
\begin{align*}
\int R(t) dt &= -\frac{\alpha(t)}{2M(t)} + kr(1 - 2r^2 + 2m^2(r^2 - 1)) \\
&\times \sqrt{\frac{3C}{(1 - r^2)(r^2 + m^2(1 - r^2))}}
\end{align*}
\]
where $C \neq 0, k \neq 0$ are arbitrary constants in Case 1-Case 2. Therefore from (3), (10), (13) and Cases 1–2, we obtain the Jacobi elliptic wave-like solutions to Eq. (6):

$$u_{21}(\xi_1) = \int \left[ \frac{k}{4M(t)} \right] \left( \frac{k^2 C_1(n^2 - 1)(n^2 - 1 - 4p^2 r^2 - 1)}{1 + \rho \csc^2 \theta} \right) dt$$

$$u_{22}(\xi_2) = \int \left[ \frac{k}{4M(t)} \right] \left( \frac{k^2 C_1(n^2 - 1)(n^2 - 1 - 4p^2 r^2 - 1)}{1 + \rho \csc^2 \theta} \right) dt$$

**Remark 2:** If we let $m \to 1$, $u_{21,2}$ contain the Solutions (8,9) in Ref.[25].

Solutions $u_{21,2}(\xi_1)(i=1,2)$ are degenerated to soliton-like solutions when the modulus $m \to 1$, and solutions $u_{21,2}(\xi_2)(i=1,2)$ are degenerated to trigonometric functions solutions when the modulus $m \to 0$.

**C. Soliton-Like and Trigonometric Function Solutions of Eq. (4) $n \neq 1, R(t) = 0$**

By balancing the highest-order linear term $v^4 v_x$ and the nonlinear $v^2 v_{xxx}$ in (11), we obtain $N = 1$ , thus we assume that (11) have the following solutions:

$$v = c_0(t) + c_1(t)e + c_2(t)f + c_3(t)g + c_4(t)h$$  (14)

with the similar process, we obtain the following solutions:

**Case 1**

$$M(t) \beta(t) = C_2 \alpha^2(t), \beta(t) = C_4 \alpha(t),$$

$$p = \pm \sqrt{1 + q^2 + \frac{k^2 C_2(1 + n)(2 + n)^2}{(1 + 2n)n^2}}$$

$$c_0(t) = c_1(t) = c_2(t) = c_4(t) = 0, m = 1,$$

$$c_3(t) = \frac{k^2 C_2(1 + n)(2 + n)}{n^2}, l = 0, r = 1,$$

$$\omega(t) = - \int \left[ \frac{k^2 C_2 \alpha(t)}{n^2} + k\gamma(t) \right] dt$$

**Case 2**

$$M(t) \beta(t) = C_2 \alpha^2(t), \beta(t) = C_4 \alpha(t),$$

$$r = \pm \sqrt{1 + q^2 + \frac{k^2 C_2(1 + n)(2 + n)^2}{(1 + 2n)n^2}}$$

$$c_0(t) = c_2(t) = c_4(t) = 0, m = 0,$$

$$c_1(t) = - \frac{k^2 C_2(1 + n)(2 + n)}{n^2}, l = 0, p = 1,$$

$$\omega(t) = - \int \left[ \frac{k^2 C_2 \alpha(t)}{n^2} + k\gamma(t) \right] dt$$

**Case 3**

$$m = 1, p^2 \neq 1, q = 1, r = l = 0, c_2(t) = c_4(t) = 0, c_3(t) = c_4(t) = 0,$$

$$\beta(t) = C_4 \alpha(t), M(t) = - \frac{n^2(1 + 2n)}{4k^2 C_2(1 + n)(2 + n)^2} \alpha(t)$$

$$c_0(t) = \frac{2k^2 C_2(1 + p)(1 + n)(2 + n)}{n^2},$$

$$c_1(t) = - \frac{2k^2 C_2(1 + n)(2 + n)(p^2 - 1)}{n^2},$$

$$\omega(t) = - \int \left[ \frac{4k^3 C_2 \alpha(t)}{n^2} + k\gamma(t) \right] dt$$

where $C_1 \neq 0, C_2, k \neq 0$ are arbitrary constants in Case 1-Case 3. Therefore from (3), (10), (14) and Cases 1–3, noticed $u = v^n$, we obtain the following soliton-like and trigonometric function solutions for Eq. (6) are expressed by

$$u_{31}(x,t) = \frac{B_{3,1}}{A_{3,1}} \cdot$$

$$B_{3,1} = \frac{k^2 C_2(1 + n)(2 + n)}{n^2} \times$$

$$\sec h[kx - \int \left( \frac{k^2 C_2 \alpha(t)}{n^2} + k\gamma(t) \right) dt]$$

$$A_{3,1} = \pm \sqrt{1 + q^2 + \frac{k^2 C_2(1 + n)(2 + n)^2}{(1 + 2n)n^2}} +$$

$$q \tanh[kx - \int \left( \frac{k^2 C_2 \alpha(t)}{n^2} + k\gamma(t) \right) dt] +$$

$$\sec h[kx - \int \left( \frac{k^2 C_2 \alpha(t)}{n^2} + k\gamma(t) \right) dt]$$

$$u_{32}(x,t) = \frac{B_{3,2}}{A_{3,2}} \cdot$$


\[ B_{32} = -\frac{k^2 C_1 (1 + n)(2 + n)}{n^2} \times \]

\[ \sec[kx - \int \left( \frac{k^3 C_1 \alpha(t)}{n^2} + k\gamma(t) \right) dt] \]

\[ A_{32} = \pm \sqrt{1 - q^2 - \frac{k^2 C_2 (1 + n)(2 + n)}{(1 + 2n)n^2}} + q \tan[kx - \int \left( \frac{k^3 C_1 \alpha(t)}{n^2} + k\gamma(t) \right) dt] \]

\[ + \sec[kx - \int \left( \frac{k^3 C_1 \alpha(t)}{n^2} + k\gamma(t) \right) dt] \]

\[ u_{33}(x,t) = \frac{2k^2 C_1 (1 + p)(1 + n)(2 + n) + B_{33}^{-1}}{n^2} \]

\[ B_{33} = \pm \frac{2k^2 C_1 (1 + n)(2 + n)(p^2 - 1)}{n^2} \]

\[ A_{33} = p + \tanh[kx - \int \left( \frac{4k^3 C_1 \alpha(t)}{n^2} + k\gamma(t) \right) dt] \]

If we let \( q = q_1 = 0 \) or \( q_2 = \pm \sqrt{1 - \frac{k^2 C_1 (1 + n)(2 + n)}{(1 + 2n)n^2}} \) and

\[ n \to p, C_2 \to \frac{\gamma}{\beta^2}, k \to p\sqrt{A}, \gamma(t) \to \alpha, \]

\[ C_1 \to \frac{1}{\beta}, \alpha(t) \to \beta \] in \( u_{31}, u_{32} \),

we have

\[ u_{31}(x,t) \to u_{31}(x,t) = \frac{\pm A(1 + p)(2 + p) \sec \left[ \sqrt{\frac{A}{p}} \left( x - (\alpha + A)t + \xi_0 \right) \right]^{-1}}{\sqrt{\beta^2 + A^2 \left( 1 + p \right)^2}} \]

\[ \left. \frac{1}{2p} \pm \sqrt{\frac{\beta^2 - A^2 \left( 1 + p \right)^2}{2p}} \pm A \left( 1 + p \right) \sec \left[ \sqrt{\frac{A}{p}} \left( x - (\alpha + A)t + \xi_0 \right) \right]^{-1} \right] \]

\[ u_{31,1}(x,t), u_{31,2}(x,t) \]

\[ u_{31,1}(x,t), u_{31,2}(x,t) \]

\[ \text{Remark 3: If we let} \]

\[ n \to p, C_2 \to \frac{\gamma}{\beta^2}, k \to p\sqrt{-A}, \gamma(t) \to \alpha, \]

\[ C_1 \to \frac{1}{\beta}, \alpha(t) \to \beta, q = 0 \]

or \( q = \pm \sqrt{1 - \frac{k^2 C_2 (1 + n)(2 + n)}{(1 + 2n)n^2}} \), \( u_{32}(x,t) \) turns to the Solutions (32)(33) in Ref.[20]. noticing that \( \tanh \xi = \frac{\sinh \xi}{\cosh \xi} \), \( u_{33}(x,t) \) contains the Solution (3.18) in Ref.[22]. Our method contains all the results mentioned by the \( G'/G \) method [25], the improved sub-ODE method [20], [21] and auxiliary equation technique [18], etc.

The properties of the new soliton-like wave solutions \( u_{31} \) and periodic-like solutions \( u_{32} \) are shown in Fig. 3 and Fig. 4.

![Fig. 3 Solution \( u_{31} \)](image)

when \( n = 2, k = q = \alpha(t) = \gamma(t) = 1, C_1 = C_2 = 1 \) and \( t = 0 \).

![Fig. 4 Solution \( u_{32} \)](image)

when \( n = 2, k = q = \alpha(t) = \gamma(t) = 1, C_1 = C_2 = 1 \) and \( t = 0 \).

**Remark 4:** All the solutions obtained in this paper for Eq. (4) have been checked by Mathematica software.

To our knowledge, the explicit solutions except \( (u_{12,1}, u_{12,4}, u_{22}) \) we obtained here to Eq. (4) are not shown in the previous literature. They are new exact solutions of Eq. (4).

**IV. CONCLUSIONS**

In this paper, we have found abundant new types of exact solutions for the variable-coefficient generalized KdV–mKdV equation by using the generalized Jacobi elliptic functions expansion method and computerized symbolic computation. More importantly, our method is quite simple and powerful to find new solutions to various kinds of nonlinear evolution equations, such as Schrodinger equation, Boussinesq equation etc. We believe that this method should play an important role for finding exact solutions in the mathematical physics.
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